Continuous Wavelet and Short-Time Fourier Transforms and Frames

"Man lives between infinitely large and infinitely small."

Blaise Pascal, *Thoughts*

Continuous wavelet transform

Continuous short-time Fourier transform

Frames of wavelets and STFT

Introduction

Expansions of continuous-time functions in terms of two variables

- WT shift and scale
- STFT shift and frequency

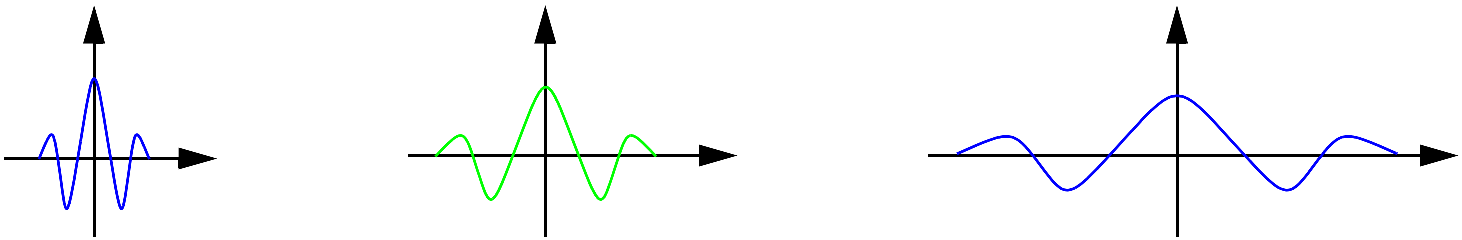
One-variable function mapped into a two-variable function

- redundant representation (1D to 2D)
- can be discretized
- leads to frames
- “critical sampling” (for example, orthonormal bases)

Continuous-time wavelet transform

Basis functions

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$



Analysis formula

$$X(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty}\psi\left(\frac{t-b}{a}\right)x(t)dt$$

Reconstruction formula

$$x(t) = \frac{1}{C_\psi}\cdot\int_{0}^{\infty}\int_{-\infty}^{\infty}X(a,b)\psi_{a,b}(t)\frac{dadb}{a^2}$$

- redundant representation 1D into 2D

Continuous-time wavelet transform
... admissibility condition

Wavelet should satisfy admissibility condition

$$C_{\psi} = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$$

In practice, if wavelet has sufficient decay this reduces to

$$\Psi(0) = 0$$

that is, wavelet has zero mean

Proof of inversion formula

- C_{Ψ} appears explicitly

Continuous wavelet transform...
... properties

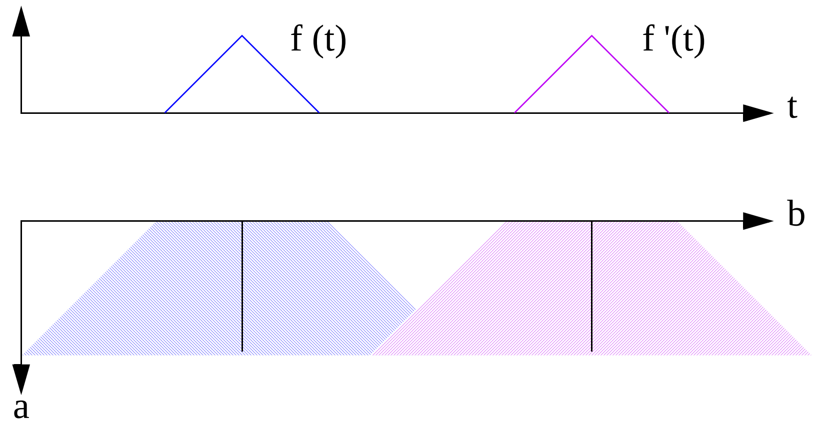
Overview

- linearity
- shift
- scale
- energy conservation
- localization in time and frequency
- characterization of regularity
- characterization of singularities
- reproducing kernel

Linearity

- $\mathrm{CWT}(x + y) = X(a, b) + Y(a, b)$
- $\mathrm{CWT}(\alpha x) = \alpha \, \mathrm{CWT}(x)$

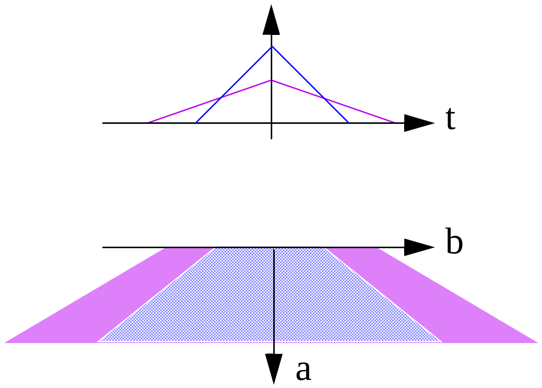
Continuous wavelet transform...
... shift property

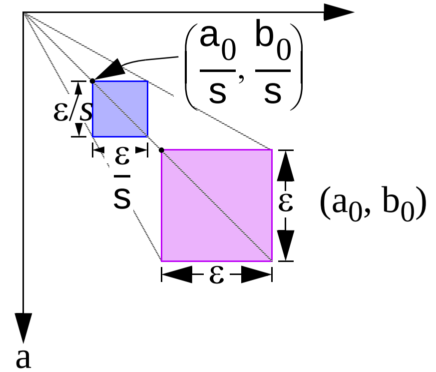


$$f'(t) = f(t - \beta)$$

$$F'(a, b) = F(a, b - \beta)$$

Continuous wavelet transform...
... scaling property





$$f'(t) = \frac{1}{\sqrt{\alpha}} \cdot f(t/\alpha)$$

$$F'(a, b) = F(a/\alpha, b/\alpha)$$

Continuous wavelet transform...
... energy conservation property

Similar to the Parseval's formula

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{C_\psi} \cdot \int_0^{\infty} \int_{-\infty}^{\infty} |X(a,b)|^2 \frac{da\,db}{a^2}$$

Generalization

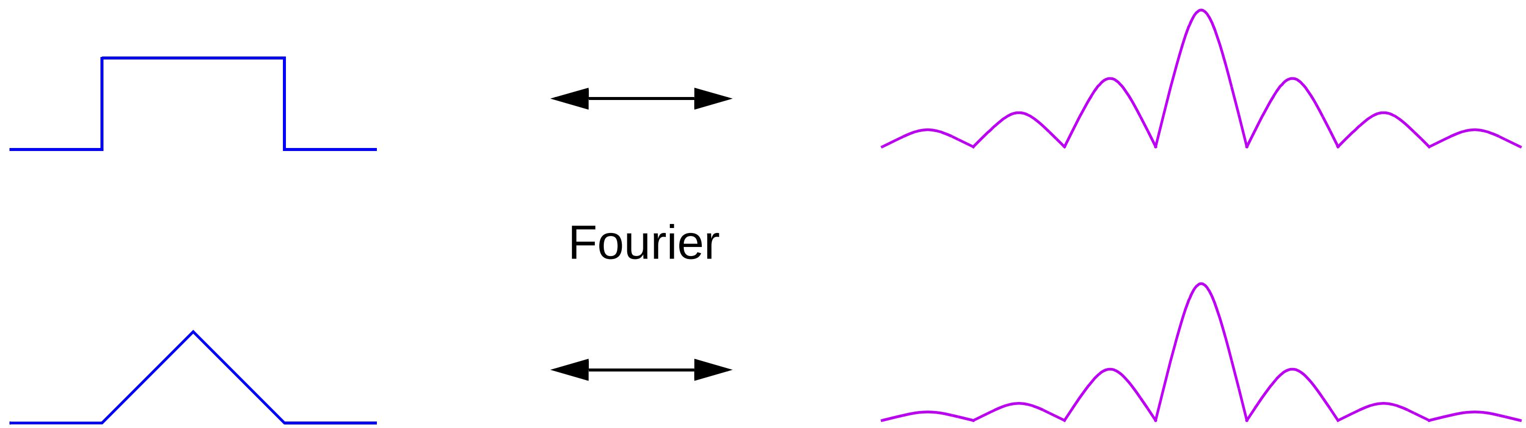
$$\int_{-\infty}^{\infty} x^*(t) y(t) dt = \frac{1}{C_\psi} \cdot \int_0^{\infty} \int_{-\infty}^{\infty} X^*(a,b) Y(a,b) \frac{da\,db}{a^2}$$

Note

- importance of the admissibility condition
- use of measure $(da\,db/a^2)$

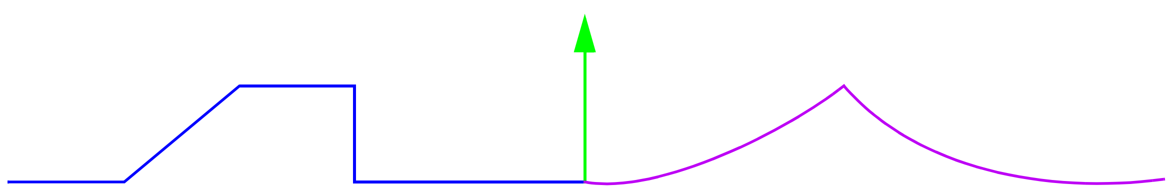
Continuous wavelet transform...
... characterization of regularity

Fourier transform: global

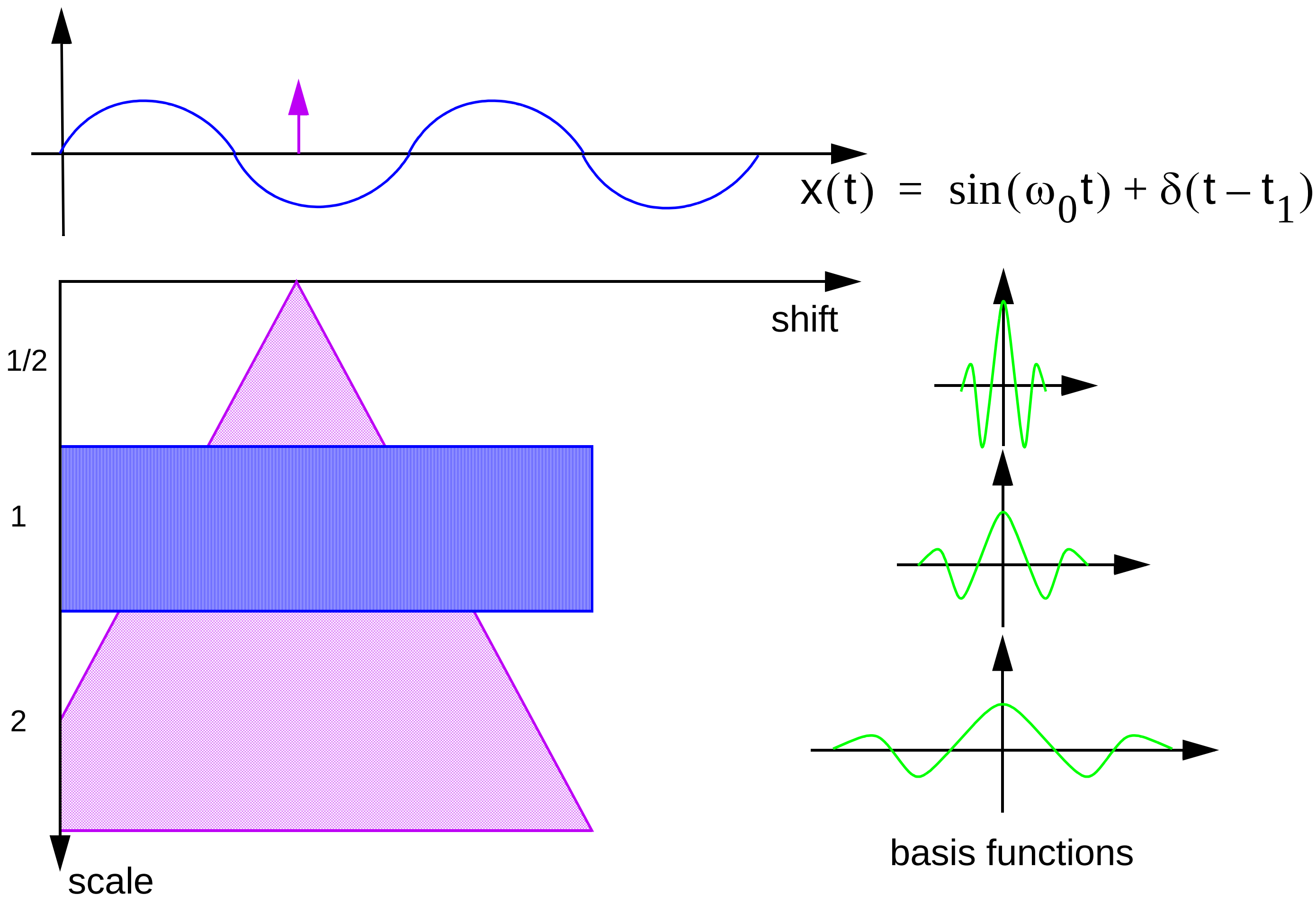


Wavelet transform: local, zooms in

Different singularities can be distinguished

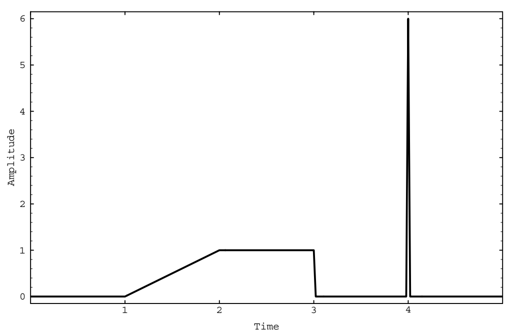


Continuous wavelet transform...
... localization

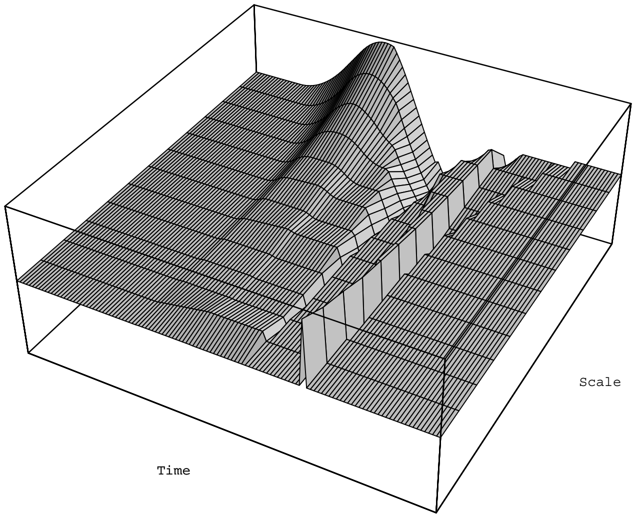


Localization of the CWT ...
... example using the Haar wavelet

signal



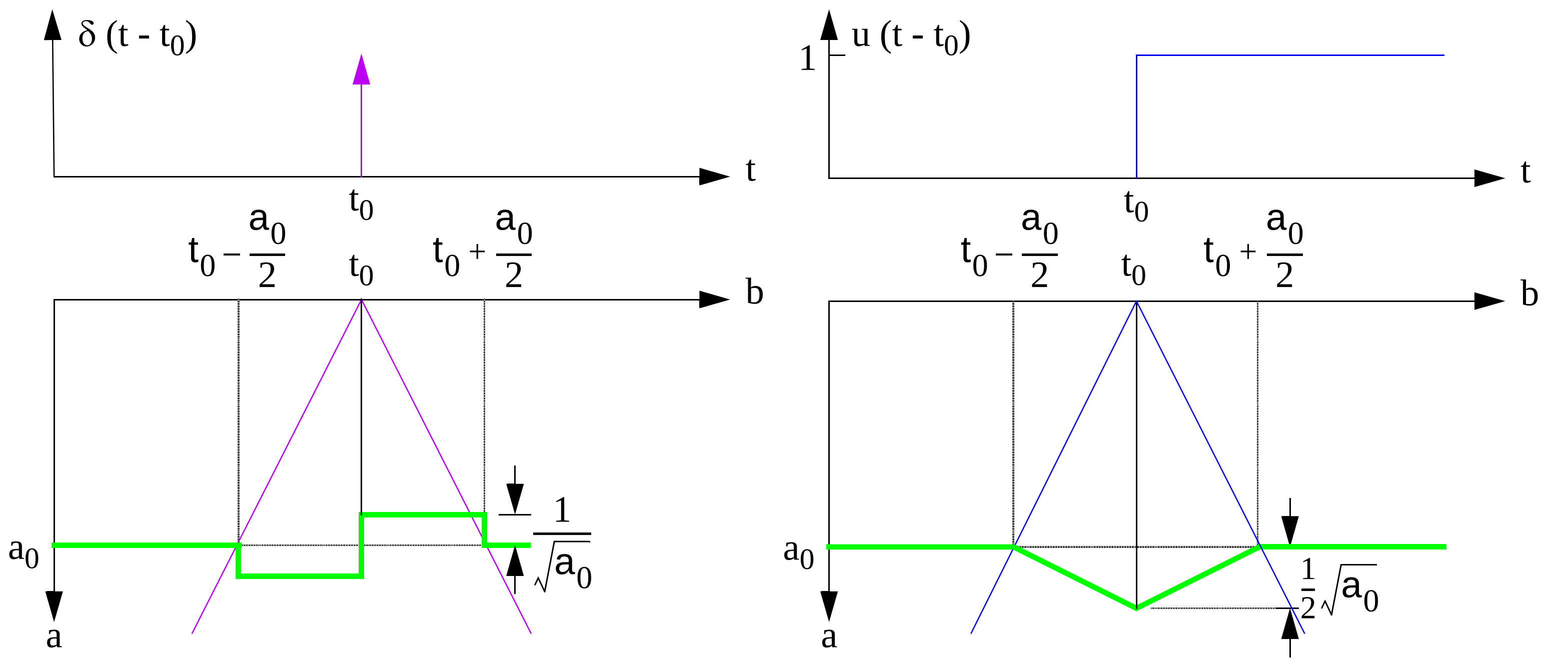
CWT



small scales
large ω
local in time

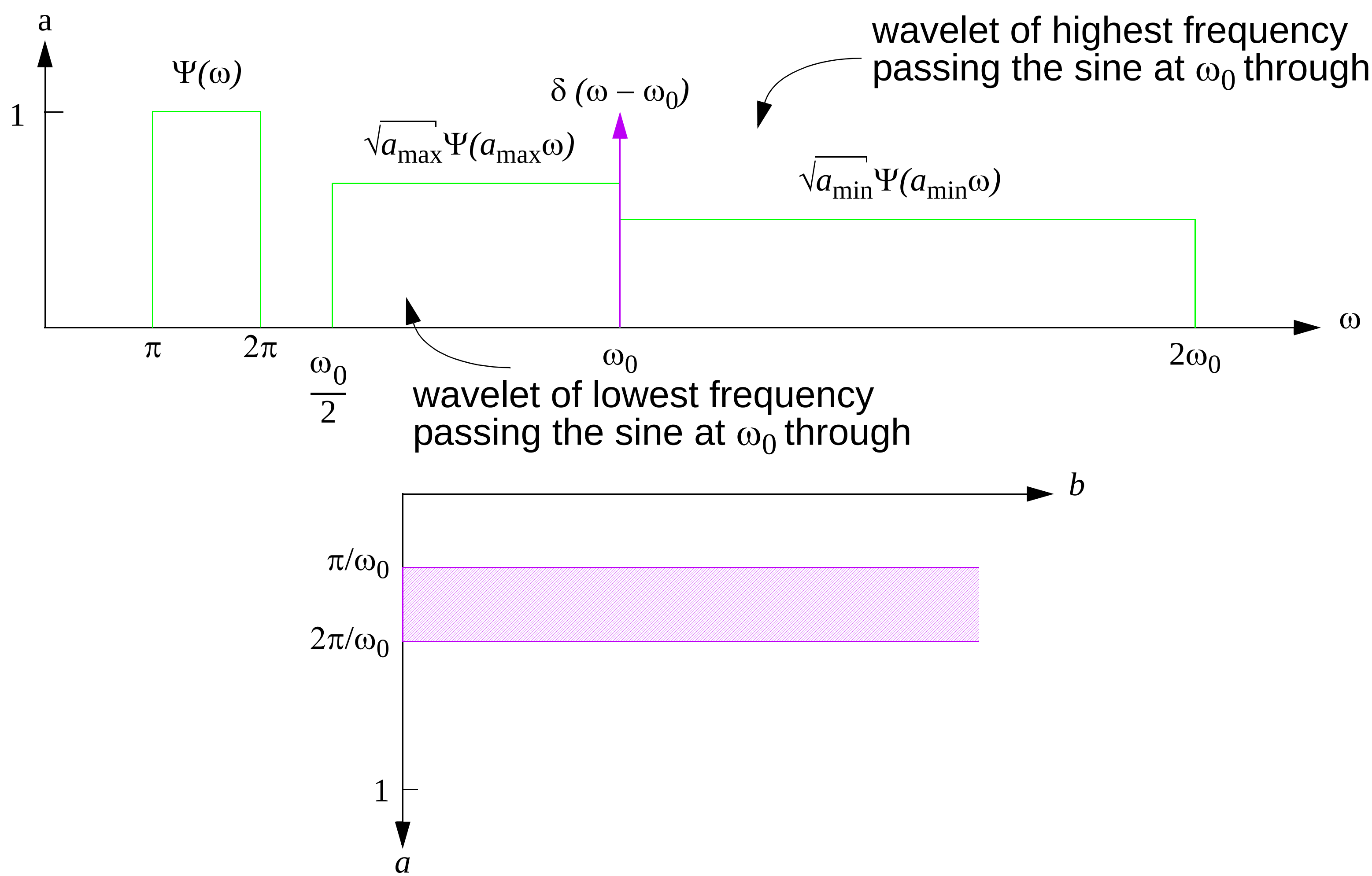
Localization of the CWT ...
... in time

Zero-phase Haar wavelet



Localization of the CWT ...
... in frequency

Sinc wavelet



Characterization of singularities

Singularity of order n

- -1: dirac
- 0: Heaviside
- 1, 2, 3... n-th derivative discontinuous

Then, the CWT transform behaves as

$$X(a, 0) = c_n \cdot a^{0.5n}$$

Thus

- singularities can be isolated
- singularity types can be distinguished

Example:

- Haar wavelet and dirac/Heaviside
- behavior as $1/\sqrt{a}$ and $\sqrt{a}$, respectively

Reproducing kernel of the CWT

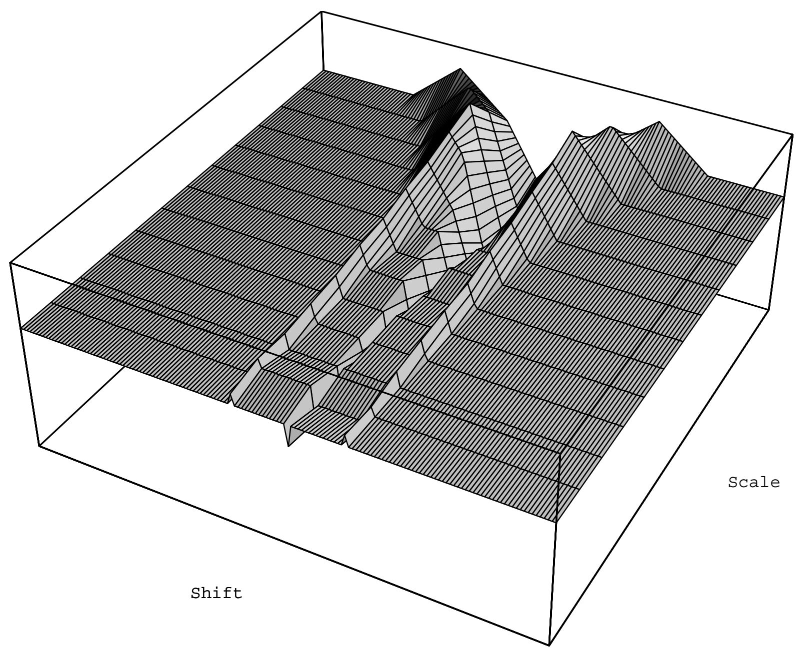
Wavelet correlation with itself across shifts and scales

$$K(a_0, b_0, a, b) = \langle \psi_{a_0, b_0}, \psi_{a, b} \rangle$$

A function F(a,b) is a CWT if and only if it satisfies

$$(a_0, b_0) = c \cdot \iint K(a_0, b_0, a, b) F(a, b) \frac{dad}{a^2}$$

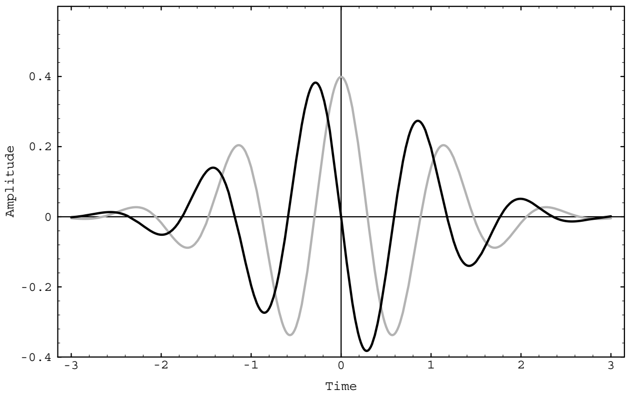
Example:



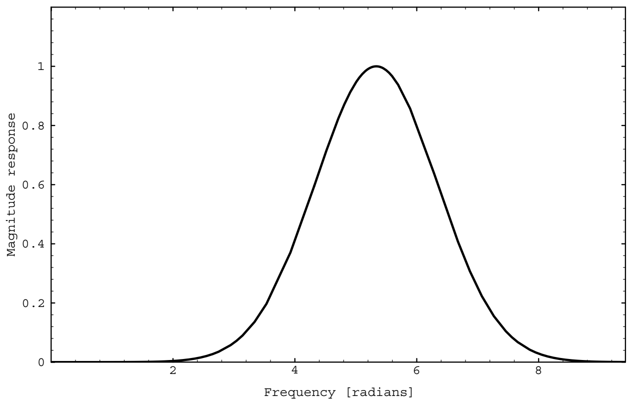
Continuous wavelet transform...
... Morlet wavelet

$$\psi(t) = \frac{1}{\sqrt{2\pi}} e^{-j\omega_0 t} e^{-\frac{t^2}{2}}$$

Not admissible but very close to it



Morlet wavelet

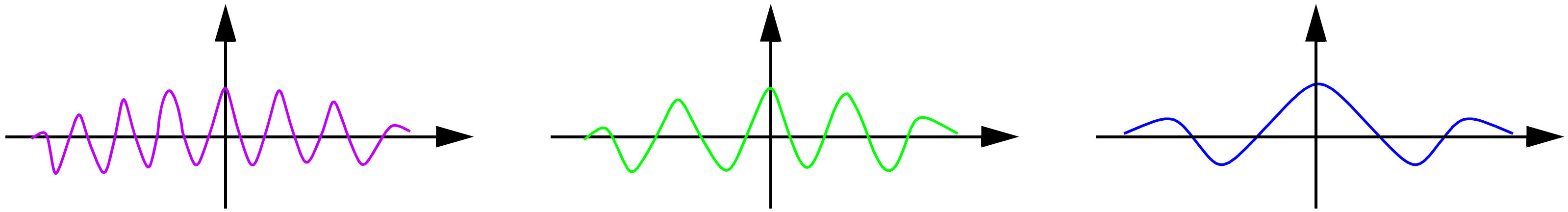


magnitude spectrum

Short-time Fourier transform (windowed Fourier, Gabor transform)

Basis functions

$$g_{\omega,\tau}(t) = e^{j\omega t}w(t-\tau) \text{ with } \|w\| = 1$$



Analysis formula

$$X(\omega,\tau) = \int_{-\infty}^{\infty} e^{-j\omega t}\overline{w}(t-\tau)x(t)dt$$

Reconstruction formula

$$x(t) = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\omega,\tau)g_{\omega,\tau}(t)d\omega d\tau$$

Gabor: used Gaussian window

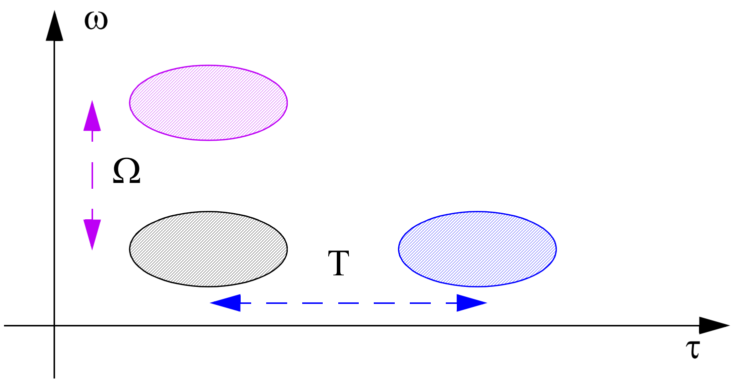
$$w(t) = \beta e^{-\alpha t^2}$$

Short-time Fourier transform...
... properties

- linearity
- shift

  if $x'(t) = x(t-T)$ then

  $$X'(\omega, \tau) = X(\omega, \tau - T)e^{-j\omega T}$$



- modulation

  if $x'(t) = e^{j\Omega t}x(t)$ then

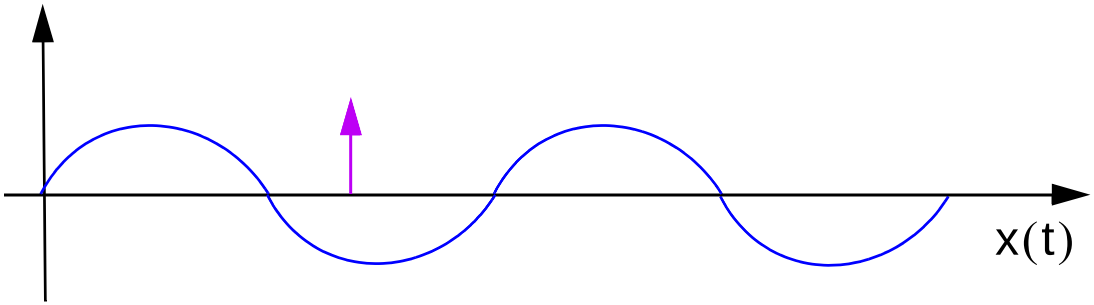
  $$X'(\omega, \tau) = X(\omega - \Omega, \tau)$$

- energy conservation

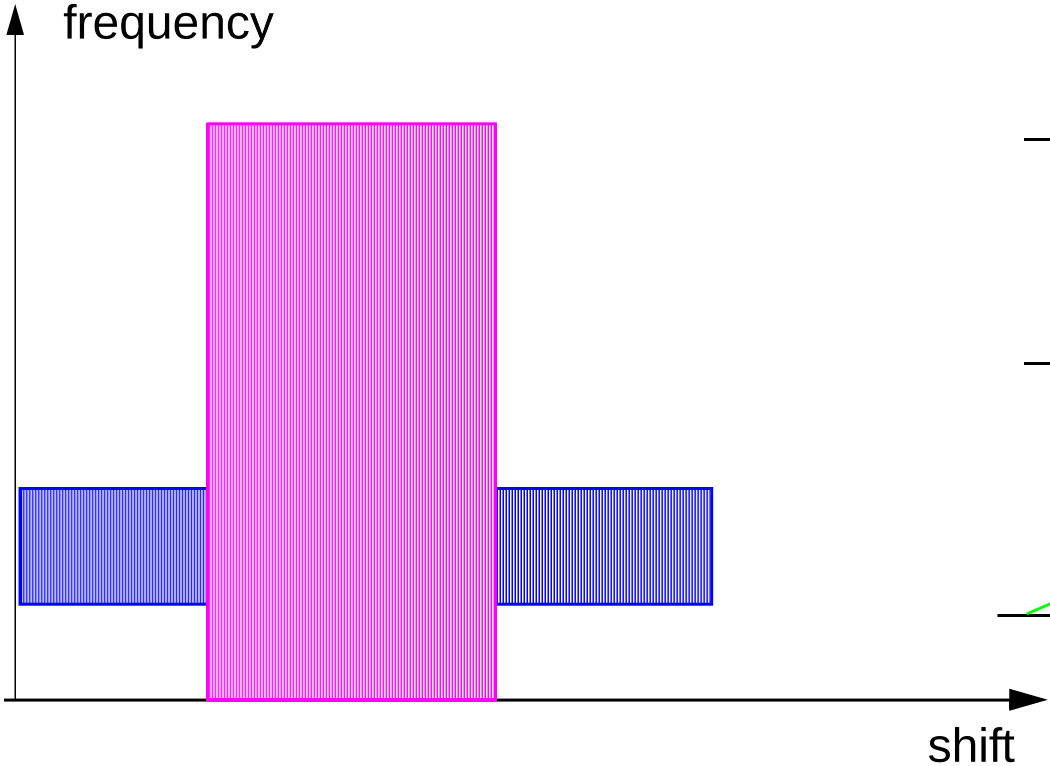
$$\|x\|^2 = \frac{1}{2\pi}\int\int|X(\omega, \tau)|^2 d\omega\, d\tau$$

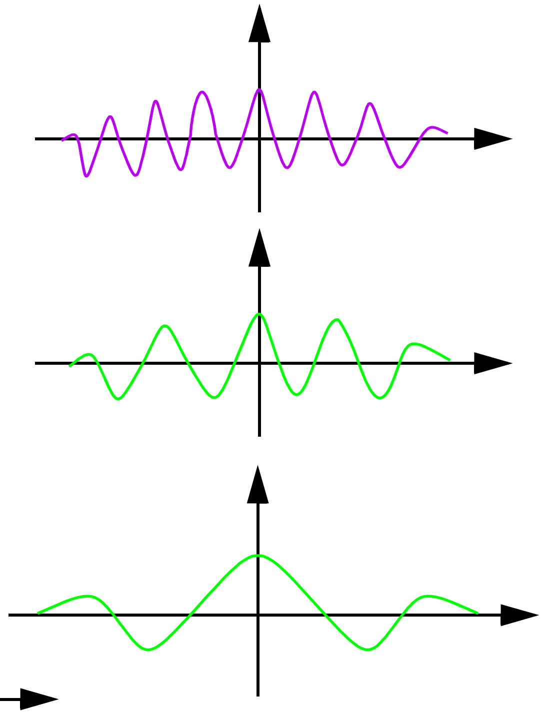
- localization in time and frequency

Short-time Fourier transform...
... localization



$$x(t) = \sin(\omega_0 t) + \delta(t - t_1)$$



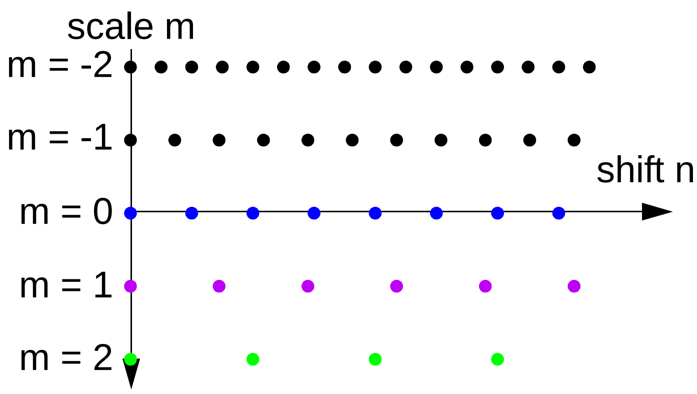


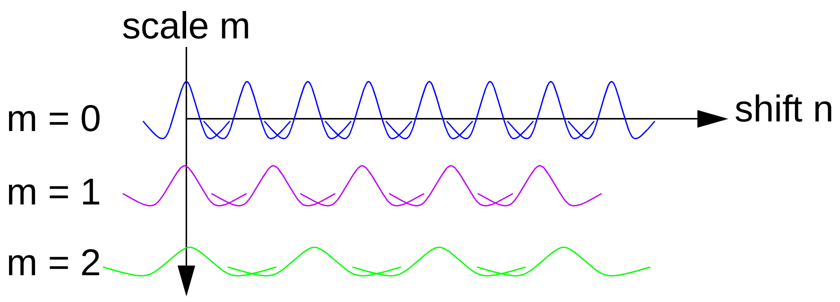
Frames of wavelets...
... discretization

Discretize scale and shift $a = a_0^m$ **and** $b = nb_0a_0^m$

- family of functions: $\psi_{m,n}(t) = a_0^{-m/2}\psi(a_0^{-m}t - nb_0)$

Example: $a_0 = \sqrt{2}$ and $b_0 = 1$



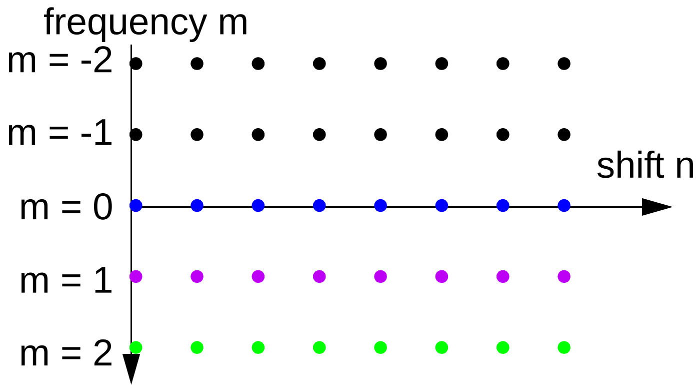


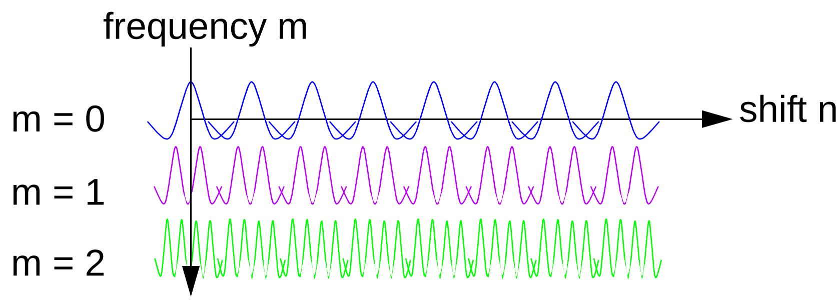
Frames of STFT...
... discretization

Discretize $\omega = m\omega_0$ **and** $\tau = nt_0$

- family of functions: $g_{m,n}(t) = e^{jm\omega_0 t}w(t - nt_0)$

Example:





Frames of wavelets and STFT ...
... can we reconstruct?

Stable reconstruction from transform coefficients

$$(\langle \psi_{m,n}, f \rangle)_{m,n} \text{ or } (\langle g_{m,n}, f \rangle)_{m,n}$$

$\psi_{m,n}$ and $g_{m,n}$ have to constitute a frame

$$A\|f\|^2 \le \sum_{m,n} \left| \langle \psi_{m,n}, f \rangle \right|^2 \le B\|f\|^2$$

Notes:

- A and B are the frame bounds
- we assume that $\|\psi_{m,n}\| = 1$
- if it were an orthonormal basis: $A = B = 1$ (Parseval)
- if $A = B > 1$: tight frame
- if $A \ll B$: bad conditioning

Frames in $\Re^N$

Family of $M > N$ nonindependent vectors covering $\Re^N$

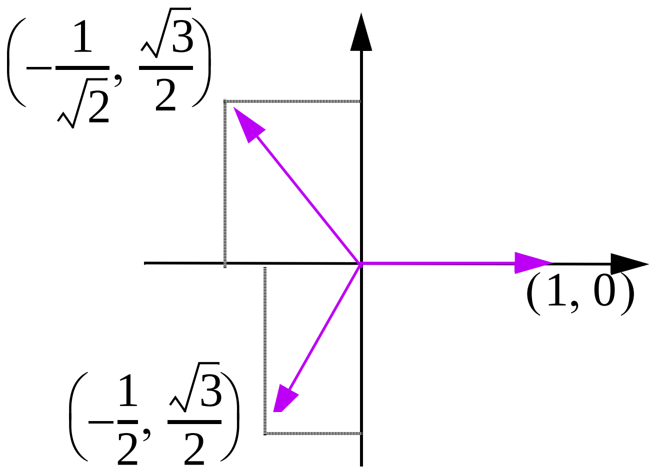
Frame: if there are A and B such that

- $0 < A \le B < \infty$
- vectors are of norm 1
- tight frame with $A = B = 1$: orthonormal basis

Example:

$$\mathbf{M} = \begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \qquad \mathbf{M}\mathbf{M}^T = \frac{3}{2}\mathbf{I}$$

$$f = \frac{2}{3}\sum_{i=0}^{2} \langle \psi_i, f\rangle \psi_i$$



Frames in $\Re^N$

Example:

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} T_{00} & T_{01} & T_{02} \\ T_{10} & T_{11} & T_{12} \\ T_{20} & T_{21} & T_{22} \\ T_{30} & T_{31} & T_{32} \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$$A \cdot \|x\|^2 \le \|y\|^2 \le B \cdot \|x\|^2$$

Inequality satisfied iff N independent rows/columns and bounded entries

Inverse: many possible, generalized inverse:

$$T^* = (T^T \cdot T)^{-1} \cdot T^T \Rightarrow T^* \cdot T = I$$

Tight Frame: $T^T \cdot T = \alpha \cdot I \Rightarrow T = \frac{1}{\alpha} \cdot T^T$

Reconstruction in frames

We need the dual frame since $f = \sum_{i=0}^{2} \langle \psi_i, f \rangle \tilde{\psi}_i$

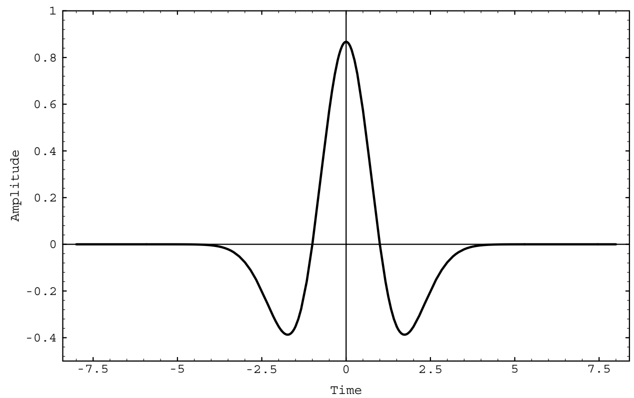
If $(\Gamma f)_i = \langle \psi_i, f \rangle$ **and** $\Gamma^{\circ} c = \sum_i c_i \psi_i$ **then**

$$\tilde{\psi}_i = \frac{2}{A+B} \sum_{k=0}^{\infty} \left(I - \frac{2}{A+B} \Gamma^{\circ} \Gamma \right)^k \psi_i$$

Frames of wavelets

Possible without harsh constraints on “mother” wavelet and a_0 and b_0

Mexican-hat function

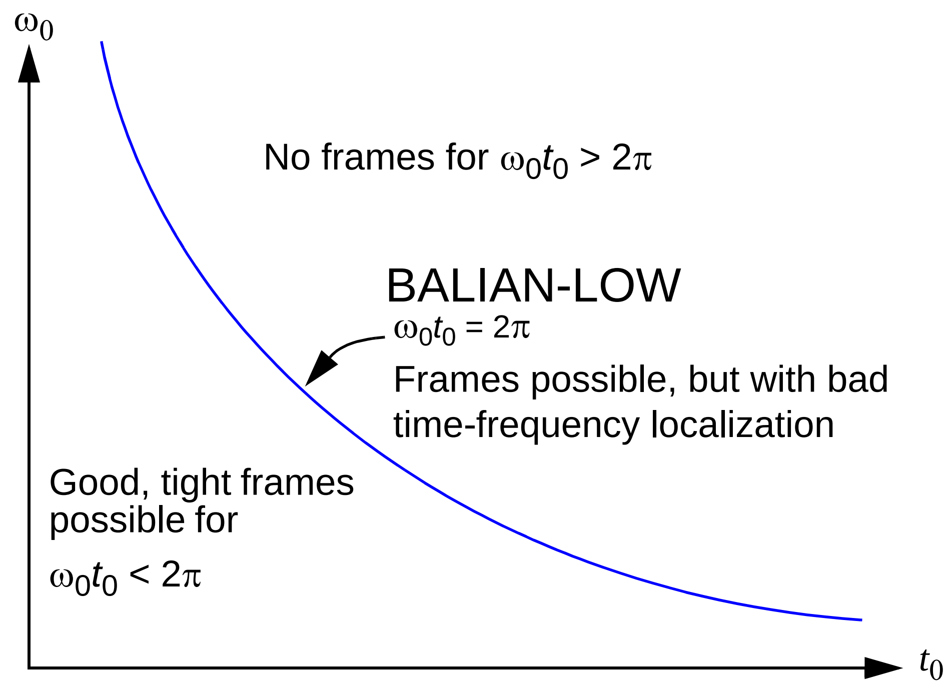


Admissibility is not a very harsh constraint in this case

Good time-frequency localization properties

Frames of STFT

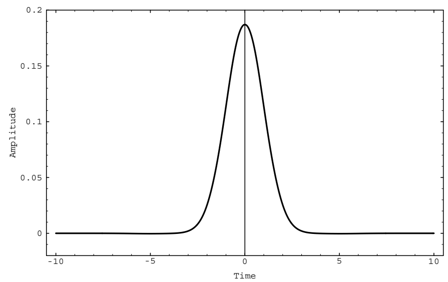
If the $g_{m,n}$ **constitute a frame then** $A \le \frac{2\pi}{\omega_0 t_0}\|g\|^2 \le B$



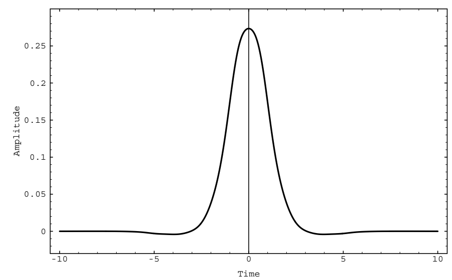
Frames of STFT ...
... example

Dual frame of STFT, Gaussian window with $\omega_0 = t_0 = \sqrt{2\pi\lambda}$

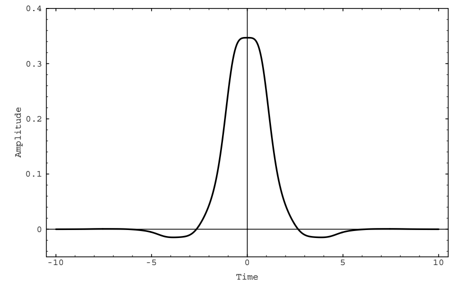
$\lambda = 0.25$



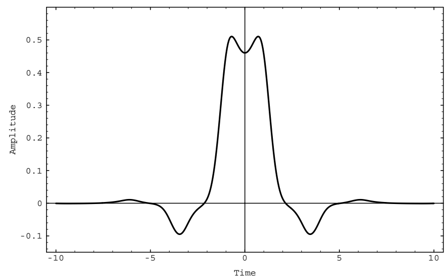
$\lambda = 0.375$



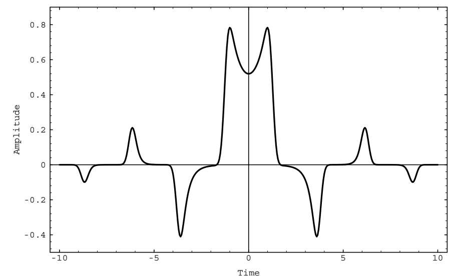
$\lambda = 0.5$



$\lambda = 0.75$



$\lambda = 0.95$



$\lambda = 1.0$