Discrete-Time Bases and Filter Banks

"What is more beautiful than the Quincunx,
which, from whatever direction you look is correct?"

Quintilian

Series expansions of discrete-time signals

Two-channel filter banks

Tree-structured filter banks

Multichannel filter banks

Pyramids and overcomplete expansions

Multidimensional filter banks

Transmultiplexers

Introduction

Focus

- series expansions of discrete-time signals
- many signals inherently discrete time
  need to manipulate them
- most signals end up being processed in discrete-time
- some construction fundamental for continuous-time wavelets
- computational tool for continuous-time signals
- computation basis for algorithms

Series expansions of discrete-time signals

Orthonormal

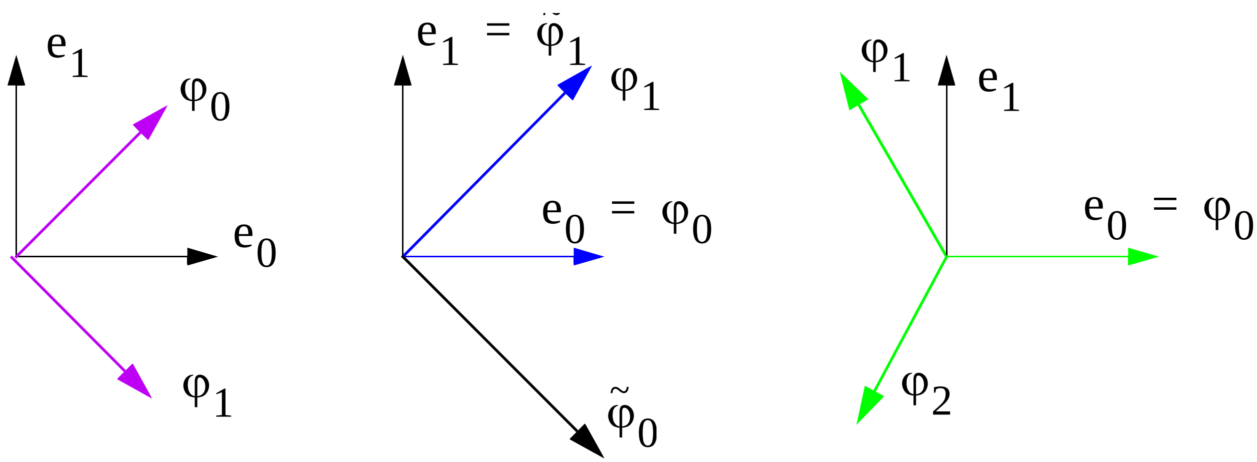
- basis functions are orthonormal
- conservation of energy

Biorthogonal

- basis functions are biorthogonal

Overcomplete

- redundant sets of functions



Series expansions of discrete-time signals...
... orthonormal

Expansion

$$x[n] = \sum_k \langle \varphi_k[l], x[l] \rangle \varphi_k[n] = \sum_k X[k] \varphi_k[n]$$

Transform coefficients

$$X[k] = \langle \varphi_k[l], x[l] \rangle = \sum_l \varphi_k^*[l] x[l]$$

Orthonormality constraint

$$\langle \varphi_k[n], \varphi_l[n] \rangle = \delta[k-l]$$

Conservation of energy

$$\|x\|^2 = \|X\|^2$$

Series expansions of discrete-time signals...
... biorthogonal

Expansion

$$x[n] = \sum_k \langle \varphi_k[l], x[l] \rangle \tilde{\varphi}_k[n] = \sum_k \tilde{X}[k] \tilde{\varphi}_k[n]$$

$$x[n] = \sum_k \langle \tilde{\varphi}_k[l], x[l] \rangle \varphi_k[n] = \sum_k X[k] \varphi_k[n]$$

Transform coefficients

$$\tilde{X}[k] = \langle \varphi_k[l], x[l] \rangle = \sum_l \varphi_k^*[l] x[l],$$
$$X[k] = \langle \tilde{\varphi}_k[l], x[l] \rangle = \sum_l \tilde{\varphi}_k^*[l] x[l]$$

Biorthogonality constraint

$$\langle \varphi_k[n], \tilde{\varphi}_l[n] \rangle = \delta[k-l]$$

Conservation of energy

$$\|x\|^2 = \langle X[k], \tilde{X}[k] \rangle$$

Series expansions of discrete-time signals ...
... why not Fourier?

Possibility: discrete-time Fourier series

- deals with limited signal space (periodic or finite)

Block discrete Fourier transform

- deals with arbitrary signals
- transform is periodically time varying
- certain time locality is achieved
- abrupt changes between intervals

Look for structured expansions

- simple characterization
- some localization in time and frequency
- certain invariance properties

Discrete-time bases from filter banks

These are structured bases for $L_2(\Re)$ where a finite set of N filters generates the basis.

Result: Given N filters with impulse responses $g_i[n]$ the set

$$\{g_0[n-Nk], g_1[n-Nk], \ldots, g_{N-1}[n-Nk]\}_{k \in Z}$$

is a basis for $l_2(\Im)$.

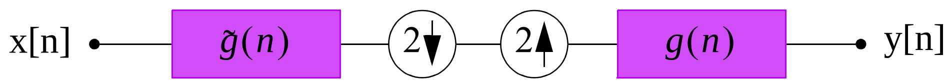
Result: Given 2 filters with impulse responses $g_i[n]$ the set

$$\{g_0[n-2k], g_1[n-2k]\}_{k \in Z}$$

is a basis for $l_2(\Im)$.

Orthogonal filter banks...
... a comprehensive example

Recall P



where $\langle g[n], g[n-2k]\rangle = \delta_k$ and $\tilde{g}[n] = g[-n]$

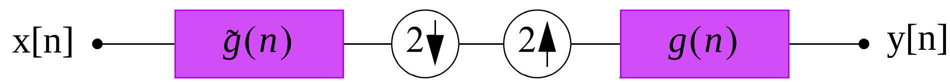
A. P **is an orthogonal projection onto the subspace** V

$$V = \operatorname{span}\{g[n-2k]\}_{k \in Z}$$

Proof: From background material and $y[n] = \sum_i \alpha_i \cdot g[n-2i]$

- $\{g[n-2k]\}_{k \in Z}$ is an orthonormal basis for V

B. Projection onto V



$$y[n] = \sum_i \alpha_i \cdot g[n-2i]$$

$$\alpha_i = \langle g[k-2i], x[k] \rangle_k$$

- convolution with $\tilde{g}[n]$ is equivalent to inner product with $g[n]$

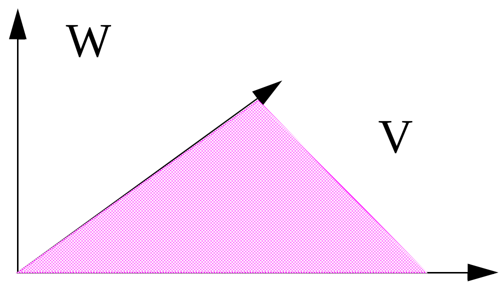
C. Orthogonal complement W to V in $l_2(\Im)$:

$$l_2(Z) = V \oplus W \qquad V \perp W$$

- find $g^{\perp}[n]$ such that

$$\langle g^{\perp}[n], g[n-2k] \rangle_n = 0 \qquad k \in Z$$

- completeness: span must be equal to $l_2(\Im)$



D. Ansatz

$$g^{\perp}[n] = (-1)^{n} \cdot g[-n+L-1]$$

- $g[n]$ is FIR of length L, L is even

Example:

$$g[n] = [a, b, c, d] \qquad g^{\perp}[n] = [d, -c, b, -a]$$

Show that:

$$\langle g^{\perp}[n], g[n-2k]\rangle_n = 0 \qquad k \in Z$$

Then:

$$\begin{bmatrix} c & d & 0 & 0 \\ a & b & c & d \\ 0 & 0 & a & b \end{bmatrix} \cdot \begin{bmatrix} d \\ -c \\ b \\ -a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

General proof: similar, use pairwise cancellation

E. Completeness:

- **Now, $g^{\perp}[n]$ and its shift by 2 are orthogonal, which follows from the orthogonality of $g[n-2k]$**

So we have an orthonormal basis for W, and we know from above that $V \perp W$. But do we have:

$$l_2(\mathfrak{I}) = V \oplus W?$$

Proof: Intuition: $l_2(\mathfrak{I})$ splits into two halves, V and W

Proof: Clean proof:

Use Parseval and show that

$$\|x\|^2 = \|x_V\|^2 + \|x_W\|^2$$

for all $x \in l_2(\mathfrak{I})$.

- this will be shown later using paraunitary matrices

F. How to find such $g[n]$'s?

$$\langle g[n], g[n-2k] \rangle_n = \delta_k$$

- this is simply the autocorrelation of $g[n]]$ evaluated at even lags, or subsampled by 2.

In z-transform domain, the autocorrelation is

$$\langle g[n], g[n-i] \rangle_n \Leftrightarrow G(z) \cdot G(z^{-1})$$

The first condition becomes equivalent to

$$\frac{1}{2}[G(z) \cdot G(z^{-1}) + G(-z) \cdot G(-z^{-1})] = 1$$

Introducing $P(z) = G(z) \cdot G(z^{-1})$ **we need**

- a symmetric positive definite polynomial $P(z)$ satisfying $P(z) + P(-z) = 2$

By spectral factorization, we get a possible $G(z)$

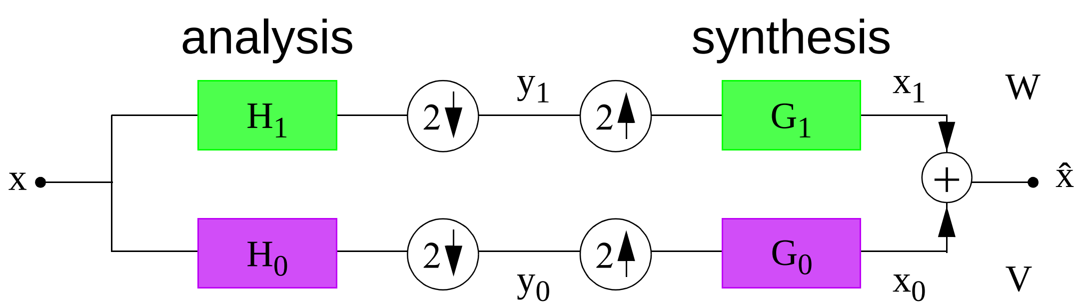
G. Putting it all together!

- $\{g[n-2k]\}_{k \in Z}$ is an orthonormal basis for V
- $\{g^{\perp}[n-2k]\}_{k \in Z}$ is an orthonormal basis for W
- $l_2(\mathfrak{I})$ splits into V and W

Use assignment:

- synthesis filters: $g_0[n] = g[n]$ $g_1[n] = g^{\perp}[n]$
- analysis filters: $h_0[n] = \tilde{g}[n]$ $h_1[n] = \tilde{g}^{\perp}[n]$

We have developed filter banks as orthonormal expansions and projections into complementary subspaces



H. Time domain view

Example: $L = 4$

$$g[n] = [a, b, c, d] \qquad g^{\perp}[n] = [d, -c, b, -a]$$

- analysis:

$$\begin{bmatrix} \dots \\ y_0[0] \\ y_1[0] \\ y_0[1] \\ y_1[1] \\ \dots \end{bmatrix} = \begin{bmatrix} a & b & c & d & 0 & 0 & 0 & 0 \\ d & -c & b & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & c & d & 0 & 0 \\ 0 & 0 & d & -c & b & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & c & d \\ 0 & 0 & 0 & 0 & d & -c & b & -a \end{bmatrix} \cdot \begin{bmatrix} \dots \\ x[0] \\ x[1] \\ x[2] \\ \dots \end{bmatrix}$$

$$y = T \cdot x$$

- synthesis: $$x = T^T \cdot y$$
- in this orthonormal case, we have:

$$T \cdot T^T = T^T \cdot T = I$$

Haar expansion and its FB implementation

$$\varphi_{2k}[n] = \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \quad n = 2k, 2k+1 \\ 0 & \text{otherwise} \end{cases}$$

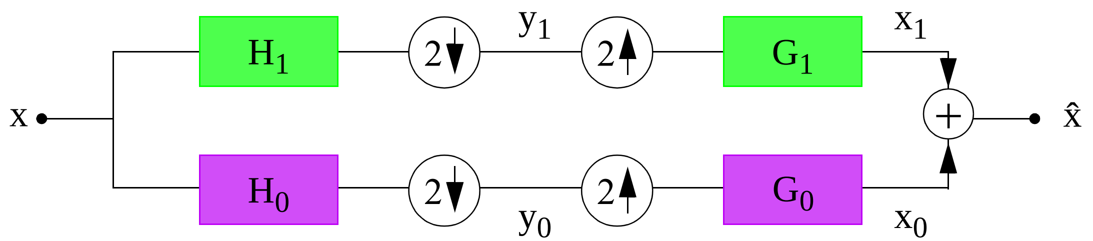
$$\varphi_{2k+1}[n] = \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \quad n = 2k \\ -\frac{1}{\sqrt{2}} & \text{for } n = 2k+1 \\ 0 & \text{otherwise} \end{cases}$$

$h_0[n] = \varphi_0[-n]$ $\quad h_1[n] = \varphi_1[-n]$

$g_0[n] = \varphi_0[n]$ $\quad g_1[n] = \varphi_1[n]$

average $\quad$ difference

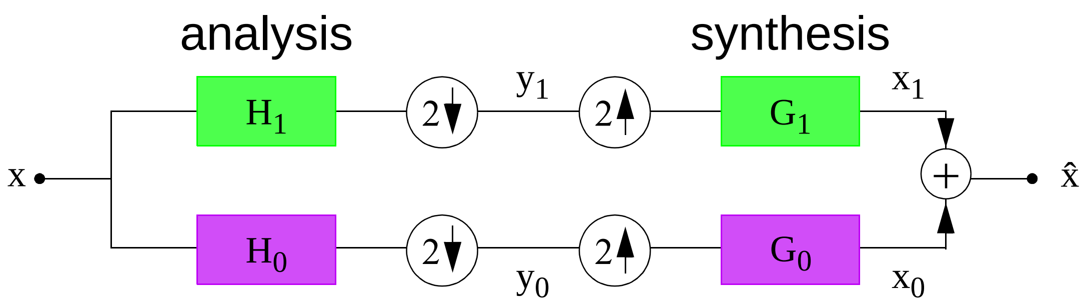
$\frac{1}{\sqrt{2}}(1, 1)$ $\quad$ $\frac{1}{\sqrt{2}}(1, -1)$



Sinc expansion and its FB implementation

$$\varphi_{2k}[n] = \frac{1}{\sqrt{2}} \frac{\sin \pi \frac{(n-2k)}{2}}{\pi \frac{(n-2k)}{2}}$$

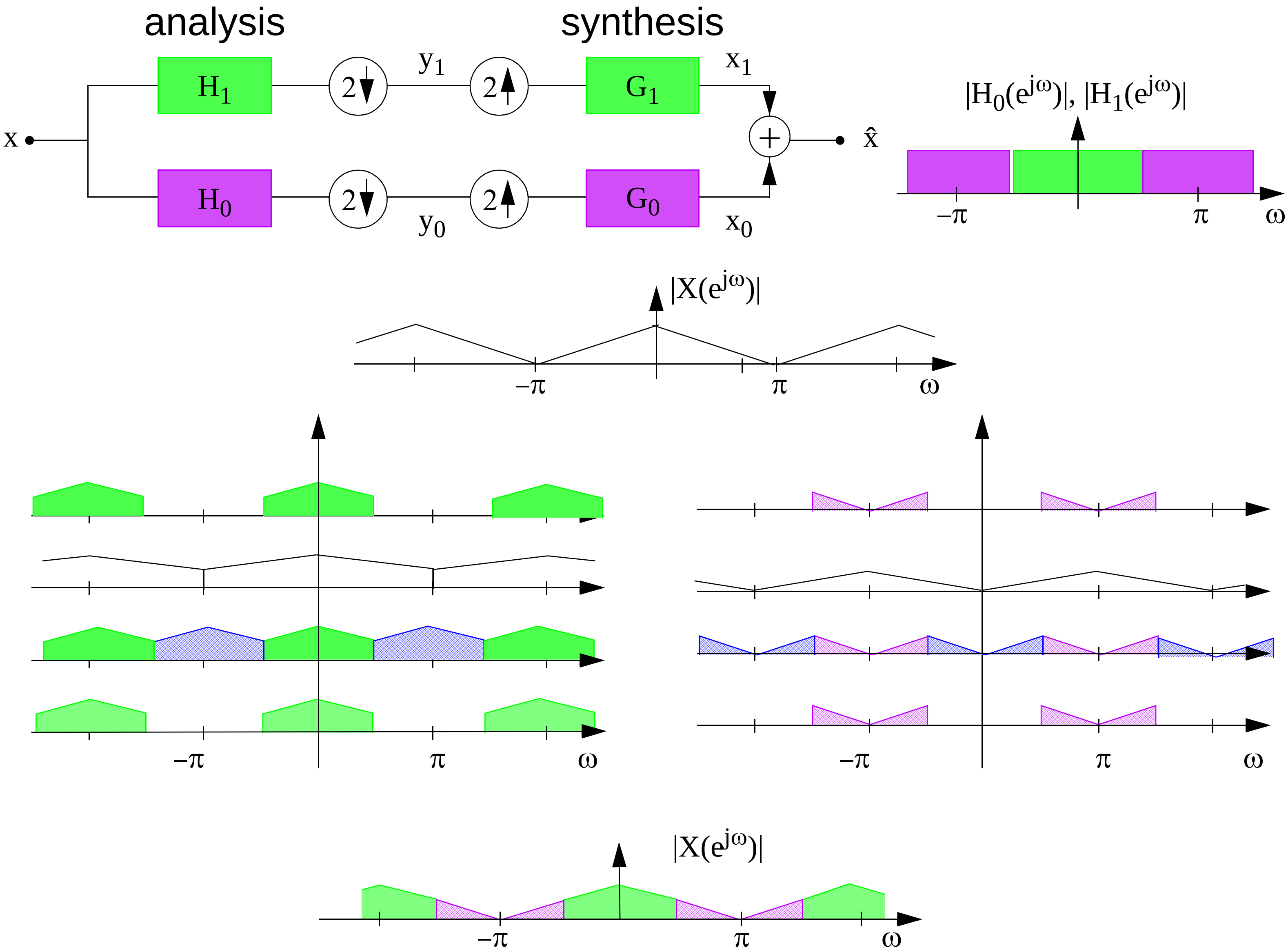
$$\varphi_{2k+1}[n] = (-1)^{n-2k} g_0[-n+2k+1]$$



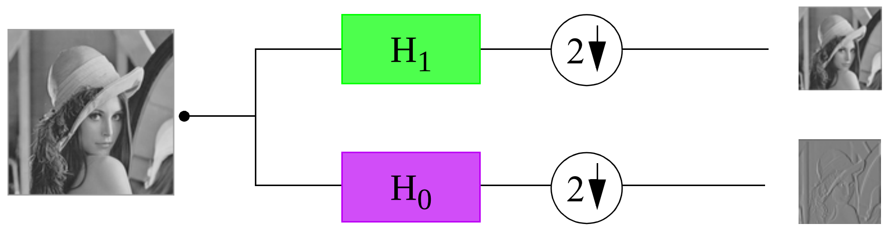
$h_0[n] = \varphi_0[-n]$ and $h_1[n] = \varphi_1[-n]$

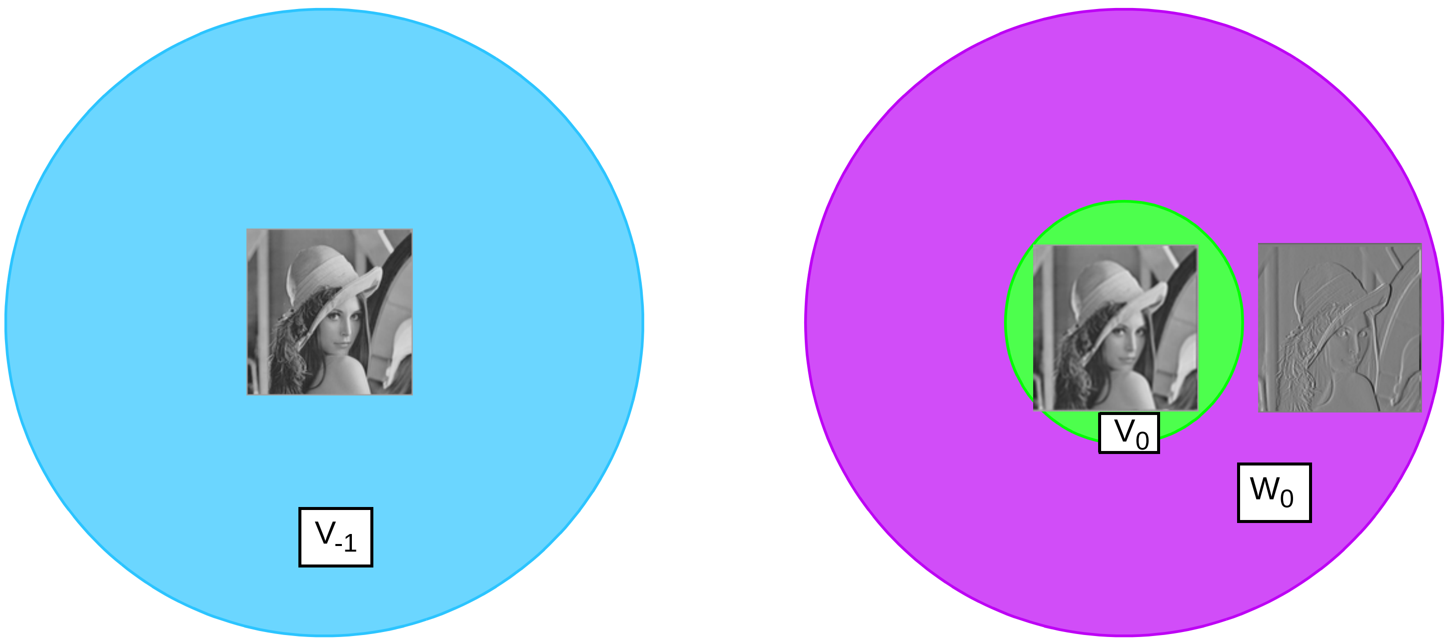
$g_0[n] = \varphi_0[n]$ and $g_1[n] = \varphi_1[n]$

Two-channel filter banks

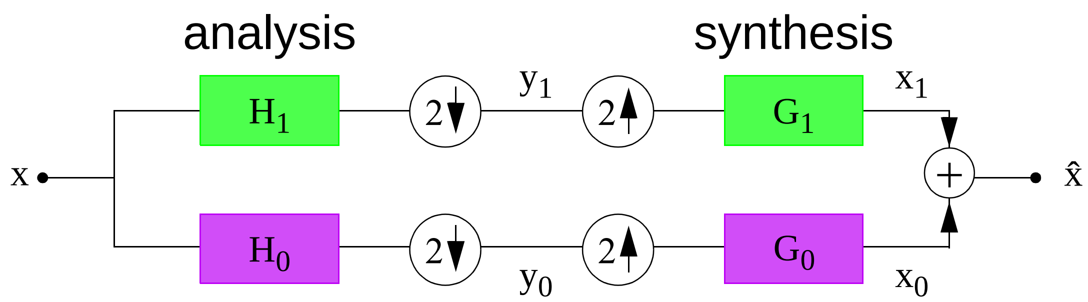


MR concept underlying filter banks



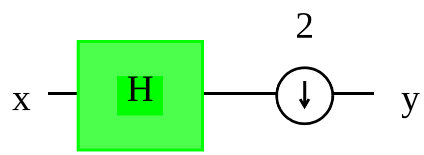


Analysis methods for filter banks



- time domain
- polyphase domain
- modulation domain

Time-domain analysis

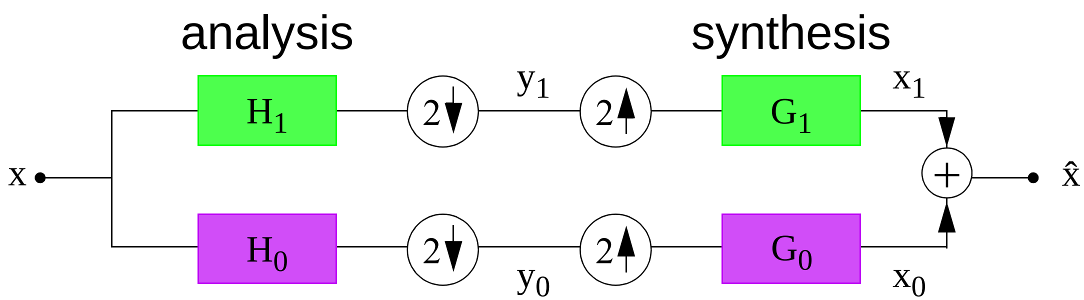


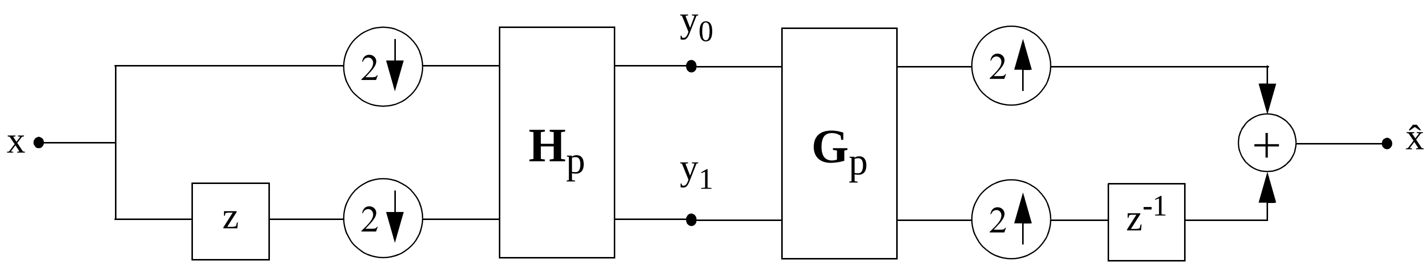
$$\begin{bmatrix} \dots \\ y[0] \\ y[1] \\ \dots \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & h[L-1] & h[L-2] & h[L-3] & h[L-4] & \dots \\ \dots & 0 & 0 & h[L-1] & h[L-2] & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \dots \\ x[0] \\ x[1] \\ \dots \end{bmatrix}$$

Perfect reconstruction: $G_0 H_0 + G_1 H_1 = I$

Orthogonal system: $(H_0)^* H_0 + (H_1)^* H_1 = I \quad G_0 = (H_1)^*$

Polyphase-domain analysis





Polyphase components: $Y(z) = IPT(z)G_p(z^2)H_p(z^2)x_p(z^2)$

- even and odd subsequences $x[2n]$ and $x[2n+1]$

Perfect reconstruction: $G_pH_p = I$

Orthogonal system: $(H_p)^*H_p = I$ $\quad G_p = (H_p)^*$

Polyphase-domain analysis...
... key results

Result: If $G_p(z)$ is paraunitary, choose:

$$H_p(z) = G_p^T(z^{-1})$$

to get perfect reconstruction.

Proof: By inspection.

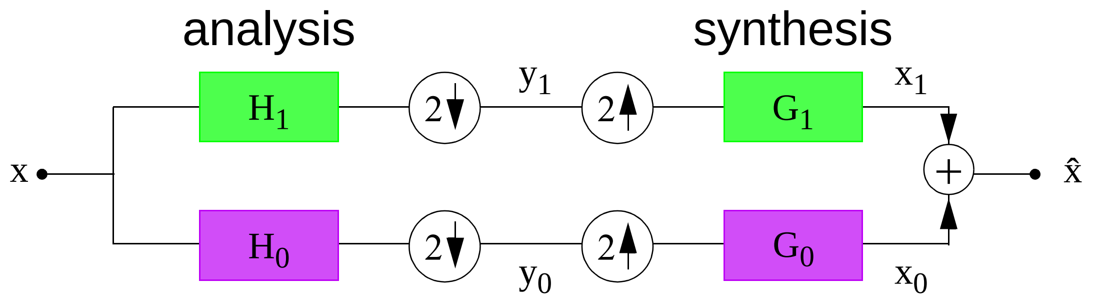
Result: $G_p(z)$ paraunitary and $H_p(z)$ as above lead to an orthonormal expansion

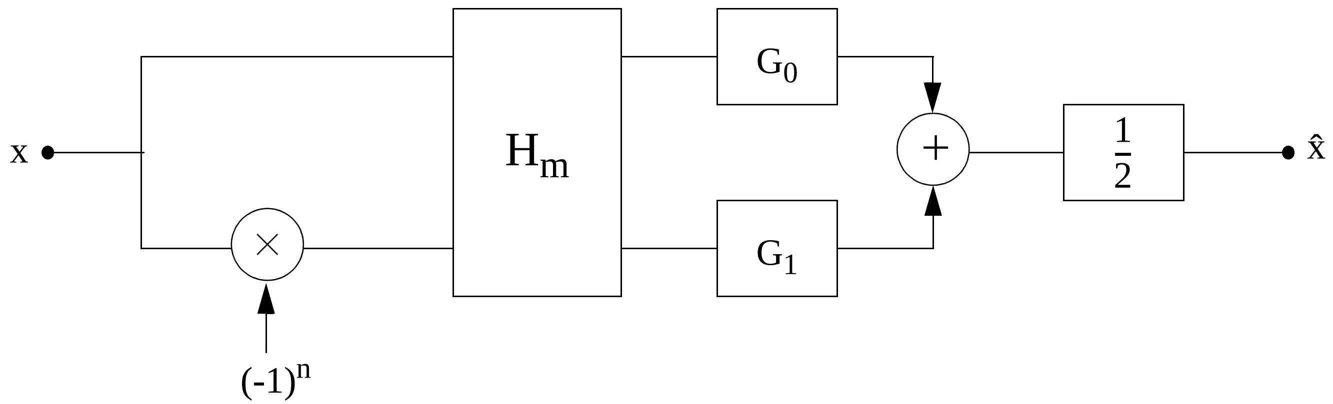
Proof: Parseval's equality.

Result: Choosing $g[n]$ orthonormal to its even translates, and the other filter as usual (ansatz) leads automatically to a paraunitary polyphase matrix

Proof: By inspection.

Modulation-domain analysis





$$Y(z) = \frac{1}{2}\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

Aliasing component: $X(-z)$

Perfect reconstruction: $GH_m = \begin{bmatrix} 2 & 0 \end{bmatrix}$

Orthogonal system: $H^* H = I$ $\quad G = H^*$

Aliasing cancellation:

I/O relationship

$$Y(z) = \frac{1}{2}\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix}\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}\begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

To guarantee aliasing cancellation

$$[G_0(z), G_1(z)] \perp [H_0(-z), H_1(-z)]$$

or

$$[G_0(z), G_1(z)] = \alpha(z) \cdot [H_1(-z), -H_0(-z)]$$

Then

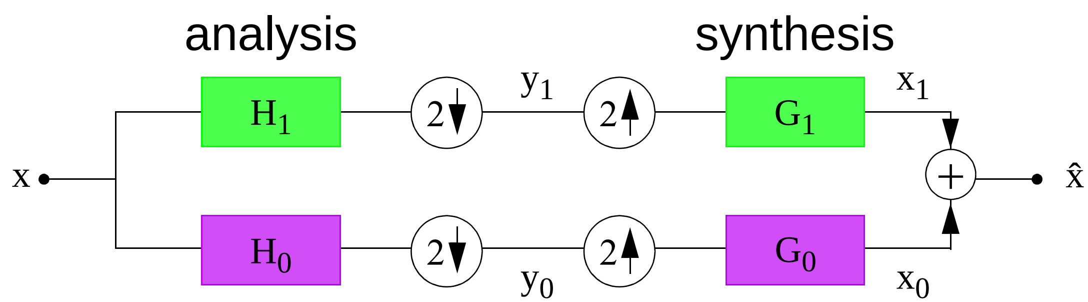
$$Y(z) = \frac{1}{2} \cdot \alpha(z) \cdot [H_0(z)H_1(-z) - H_0(-z)H_1(z)] \cdot X(z)$$

that is, $X(-z)$ is automatically cancelled, independent of the filters $H_0(z)$, $H_1(z)$ used.

Quadrature Mirror Filters (QMF)

Earliest example of aliasing cancellation, very important in speech compression

[Esteban-Galand, 1976]



Thus

$$Y(z) = \frac{1}{2} \cdot \begin{bmatrix} H(z) & -H(-z) \end{bmatrix} \cdot \begin{bmatrix} H(z) & H(-z) \\ H(-z) & H(z) \end{bmatrix} \cdot \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

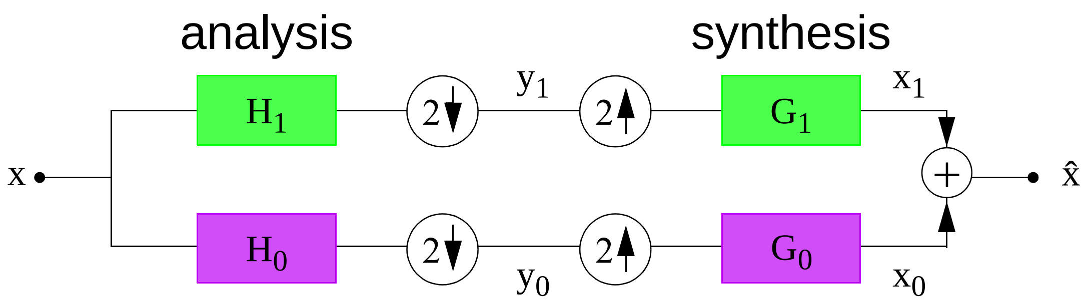
or

$$Y(z) = \frac{1}{2} \cdot [H^2(z) - H^2(-z)] \cdot X(z)$$

- $H(z)$ should be linear phase and even length

Orthogonal FIR filter banks

- synthesis filters the same as analysis (within reversal)
- **no linear phase** in this case (except trivial Haar filters)

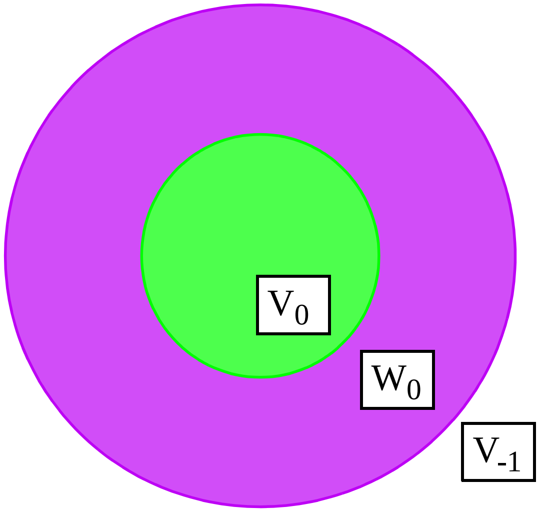


Split the space in two:

- coarse approximation and added detail
- $V_{-1} = V_0 \oplus W_0$ and $V_0 \perp W_0$

V_{-1}: **space of sequences** $l_2(\mathfrak{I})$

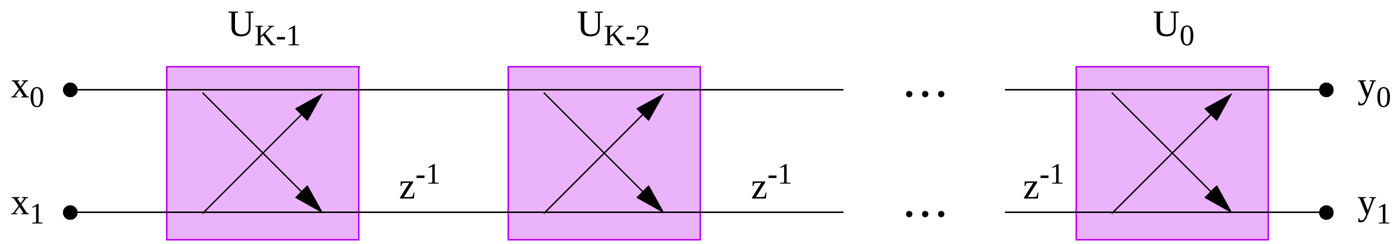
- rows of matrix H_0: basis for V_0
- rows of matrix H_1: basis for V_1
- rows of H_0 and H_1: basis for V_{-1}



Design methods for orthogonal filter banks...
... based on lattice structures

[Vaidyanathan & Hoang]

- well-conditioned
- factorization of a paraunitary matrix



$$H_p(z) = R_0 \cdot \prod_{i=1}^{N-1} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \cdot R_i \quad \text{, with} \quad R_i = \begin{bmatrix} \cos(\alpha_i) & -\sin(\alpha_i) \\ \sin(\alpha_i) & \cos(\alpha_i) \end{bmatrix}$$

- all orthogonal solutions can be reached using this cascade

Design methods for orthogonal filter banks...
... based on spectral factorization

- numerically ill-conditioned
- like “taking a square root”

[Smith&Barnwell]

- PR condition for orthogonal filter banks

$$H_0(z)\tilde{H}_0(z) + H_0(-z)\tilde{H}_0(-z) = P(z) + P(-z) = 2$$

- even subsequence is $\delta[n]$
- find such an autocorrelation sequence $P(z)$
- factor into roots $(\alpha, \frac{1}{\tilde{\alpha}})$ and assign from each pair one to $H(z)$

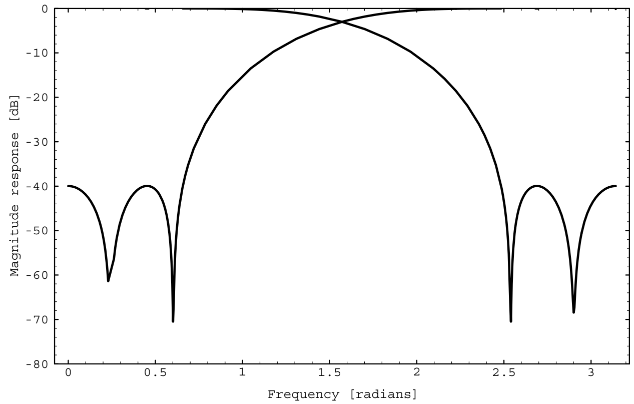
Design methods for orthogonal filter banks...
... based on spectral factorization

[Daubechies]: **minimum phase, designed to obtain wavelets**

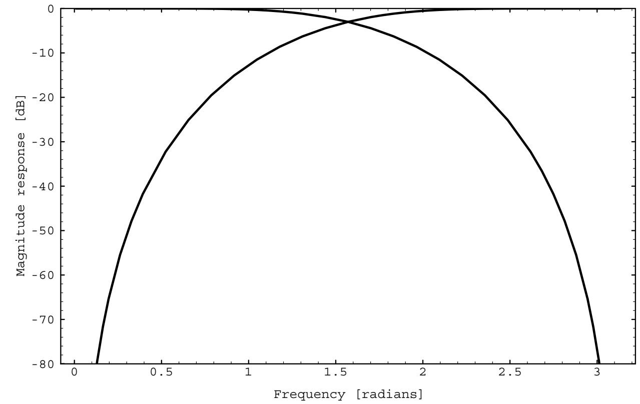
- repeat the Smith & Barnwell procedure
- impose maximum number of zeros at π
- for the rest, take factors only inside unit circle

Consequently, these filters are often called "discrete wavelets"

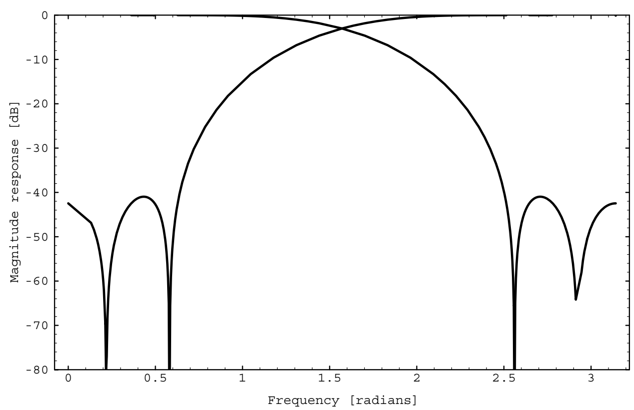
Design methods for orthogonal filter banks ...
... example of length 8



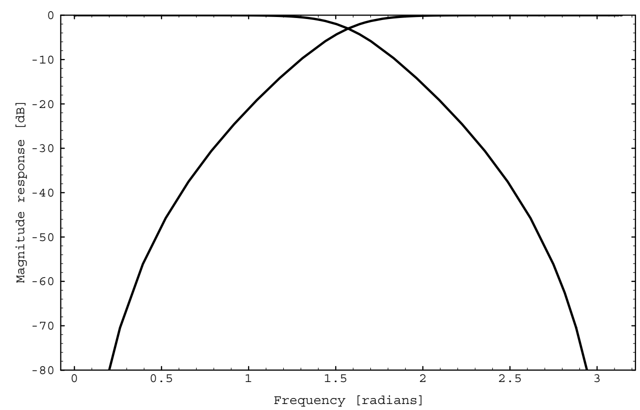
Smith & Barnwell



Daubechies



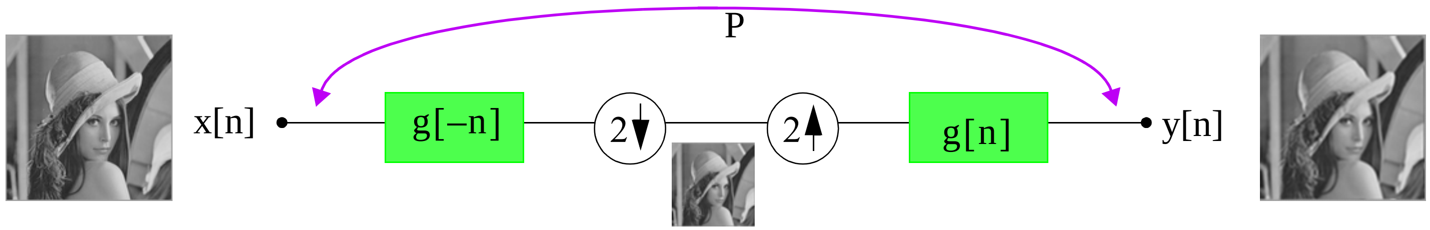
Vaidyanathan & Hoang



Butterworth
N = 4

Orthogonal filter banks ...
... projection view

Recall



with $\langle g[n], g[n-2k] \rangle = \delta_k$, then

P is an orthogonal projection

or, in other words,

P maps $l_2(\mathfrak{I})$ into V subspace of $l_2(\mathfrak{I})$

Design methods for biorthogonal filter banks

Analysis and synthesis filters are not the same anymore

Particular case of interest: linear phase

[Vetterli & LeGall]: **lattice structure**

$$H_p(z) = R_0 \cdot \prod_{i=1}^{N-1} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \cdot R_i \quad \text{, with} \quad R_i = \begin{bmatrix} 1 & \alpha_i \\ \alpha_i & 1 \end{bmatrix}$$

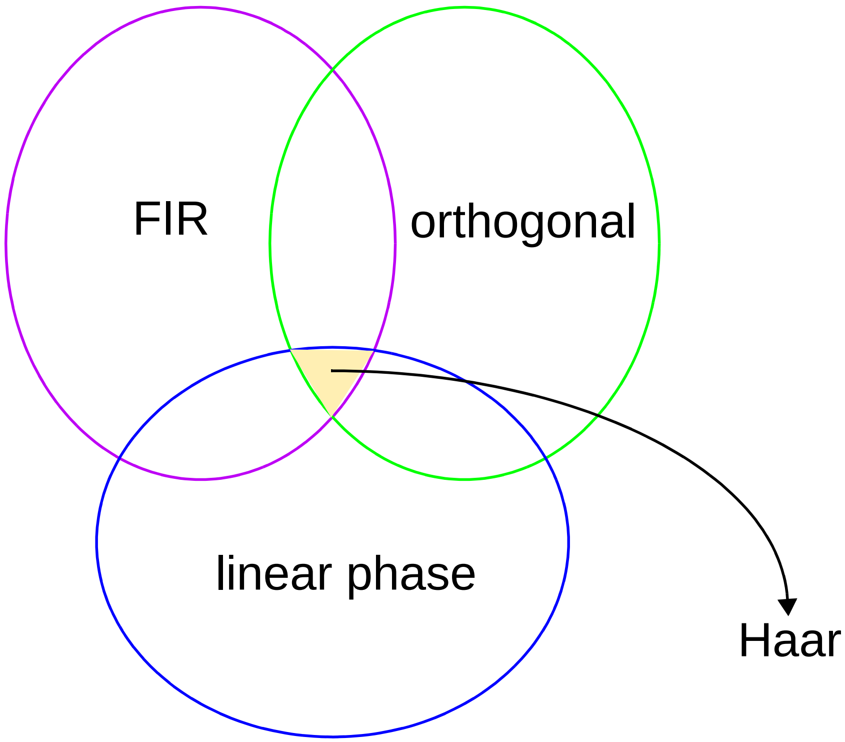
[Vetterli & Herley]:

- PR condition for biorthogonal filter banks

$$H_0(z)\tilde{H}_1(z) + H_0(-z)\tilde{H}_1(-z) = P(z) + \tilde{P}(z) = 2$$

- find such a $P(z)$
- factor into LP factors and distribute between $H_0(z)$ and $H_1(z)$

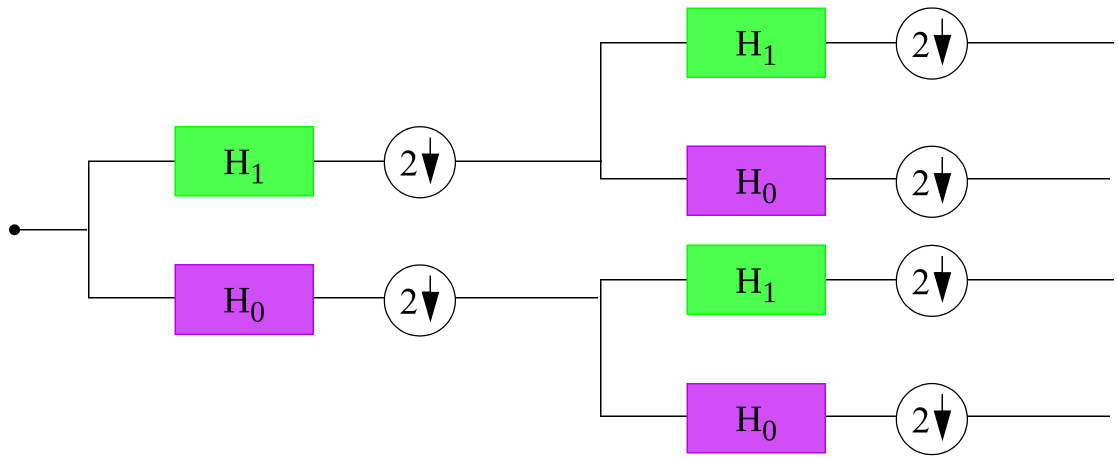
Two-channel filter banks ...
... summary



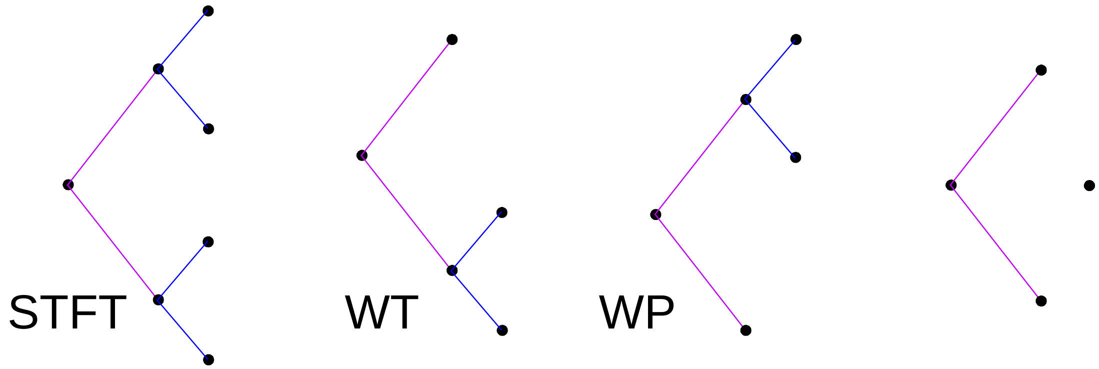
Tree-structured filter banks

Easiest way to build multichannel filter banks

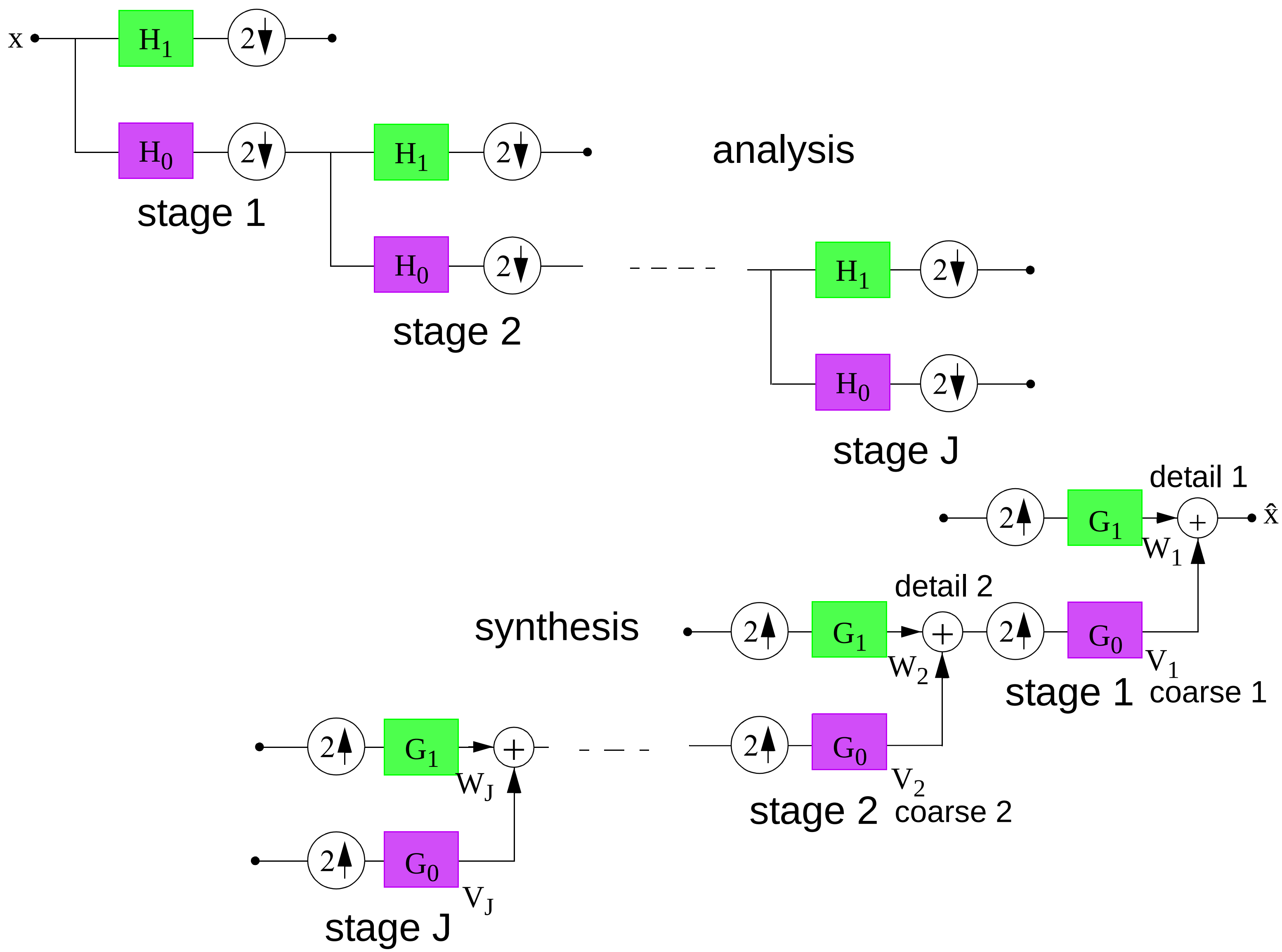
- DWT octave band
- wavelet packets



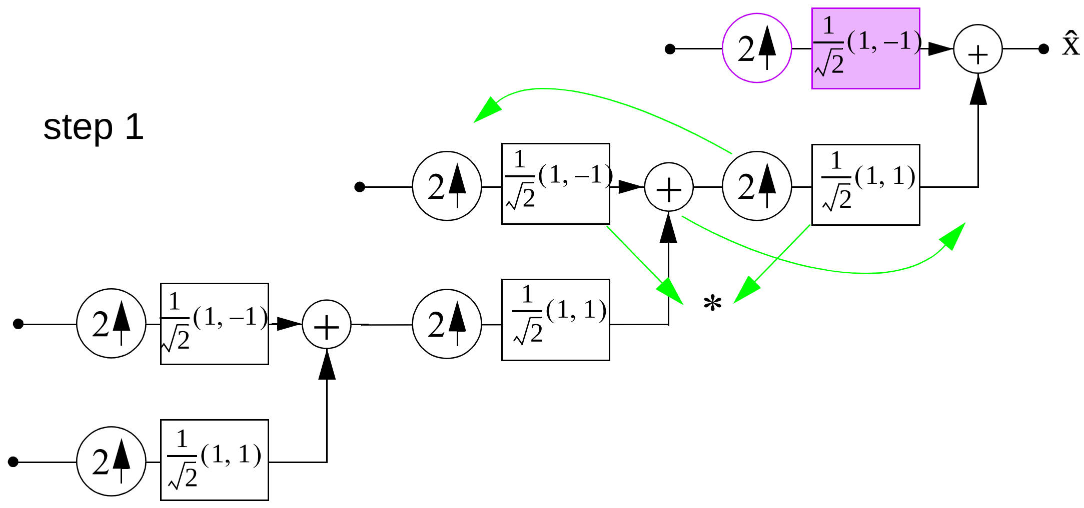
Example: All possible trees of depth two

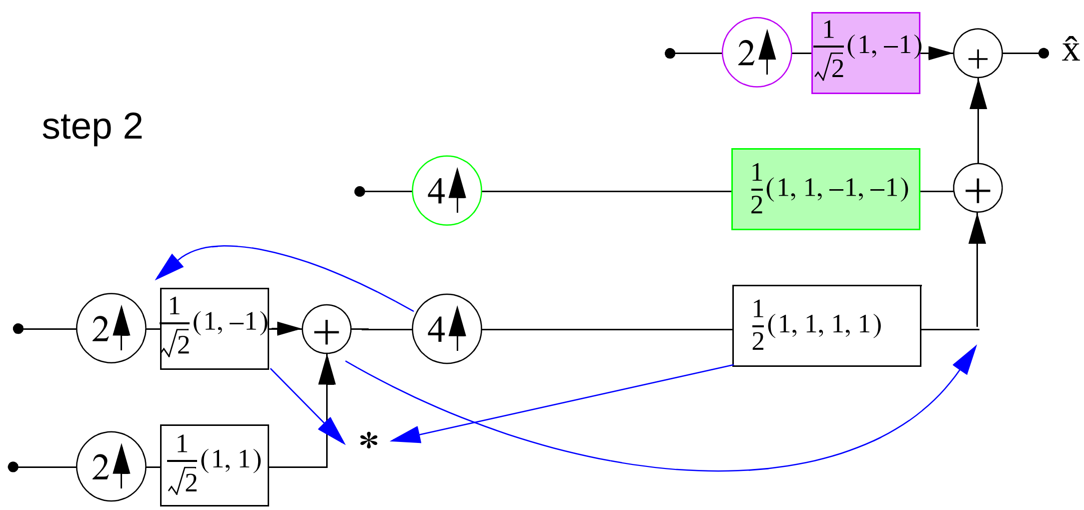


DWT - Octave band filter banks

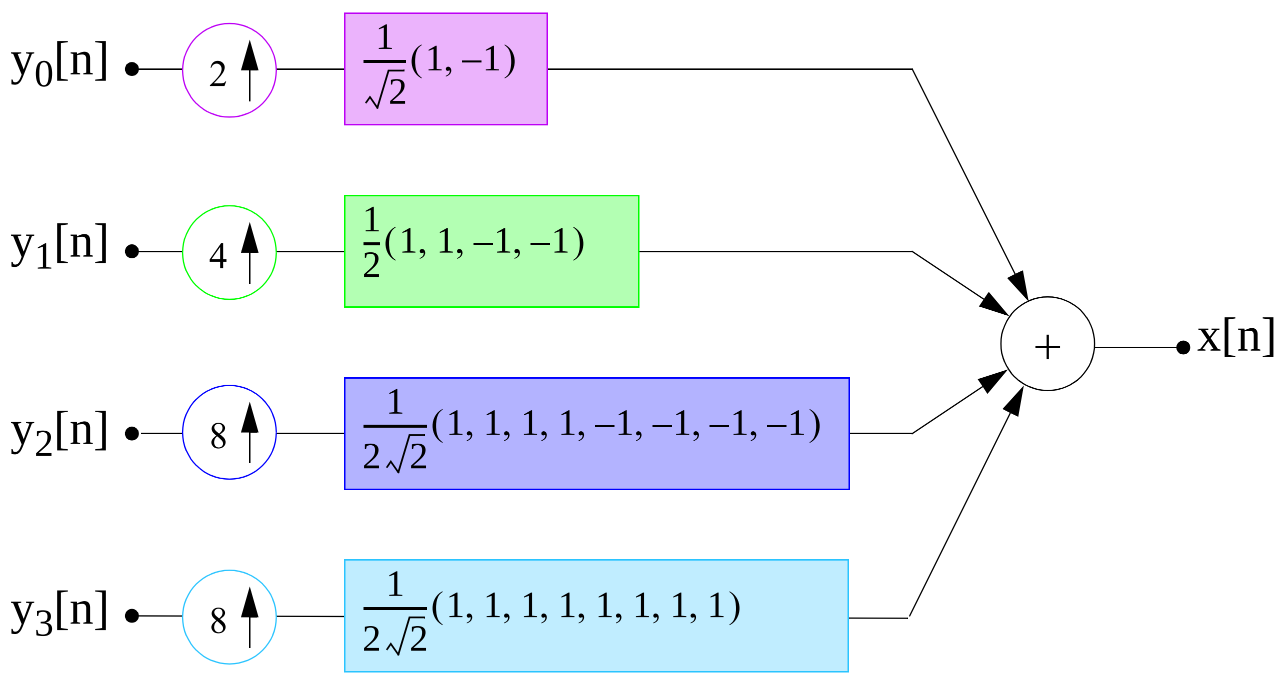


DWT - Octave band filter banks...
... three stages with Haar filters





DWT - Octave band filter banks...
... three stages with Haar filters



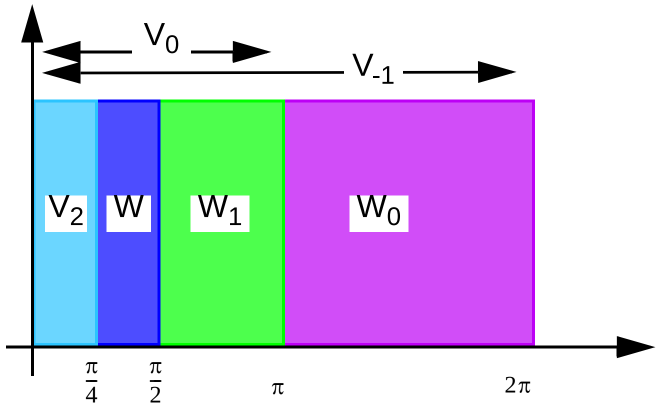
Basis functions...
... three stages with Haar filters

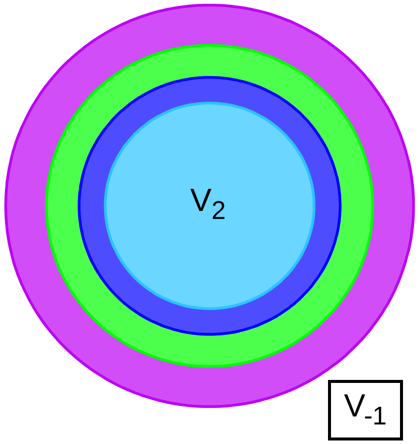
$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

- we did not include scale factors to make rows of unit energy

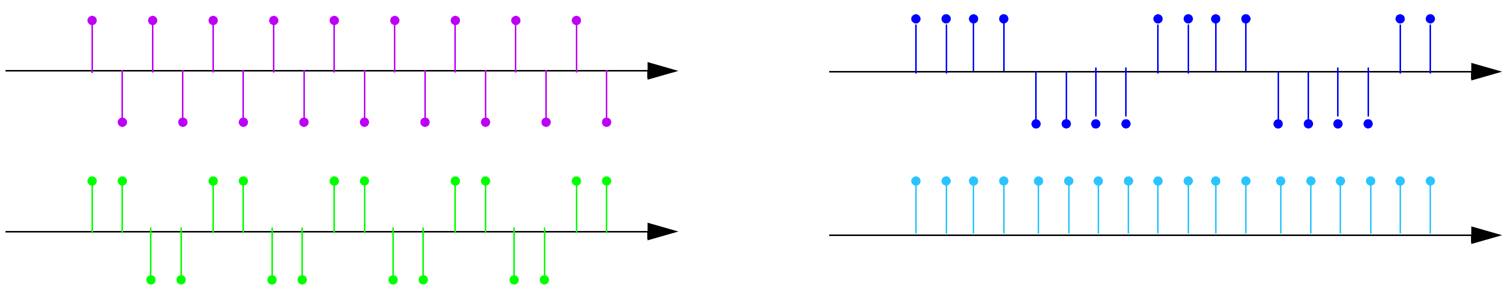
DWT - Octave band filter banks

Results in the discrete wavelet transform



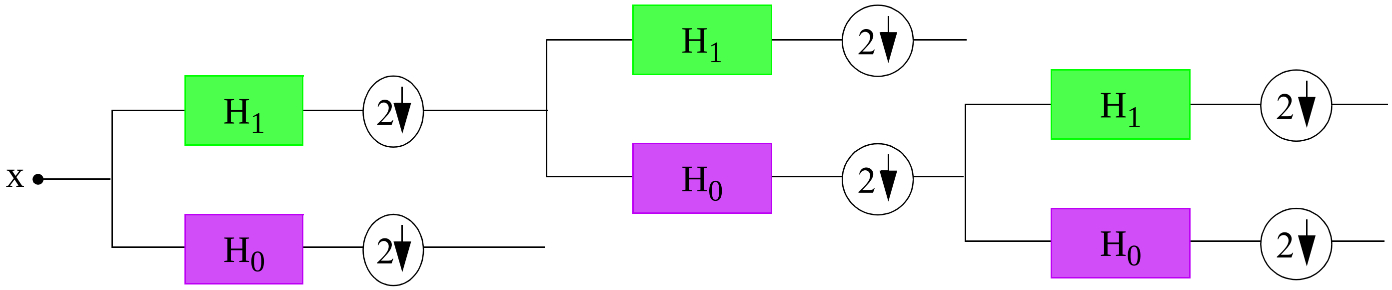


Basis functions

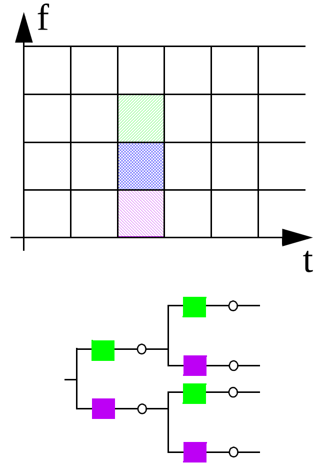


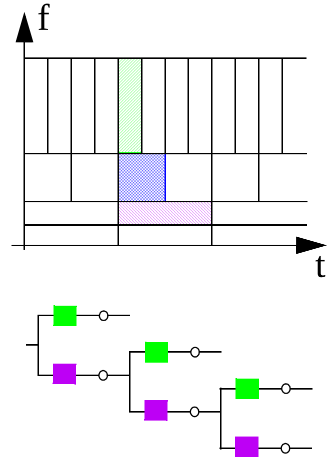
Wavelet packets

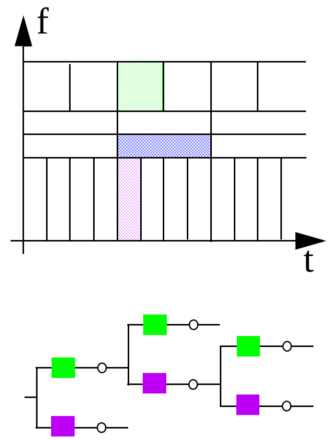
Arbitrary tree: grows to fit the signal



Tree-structured filter banks ...
... time-frequency analysis



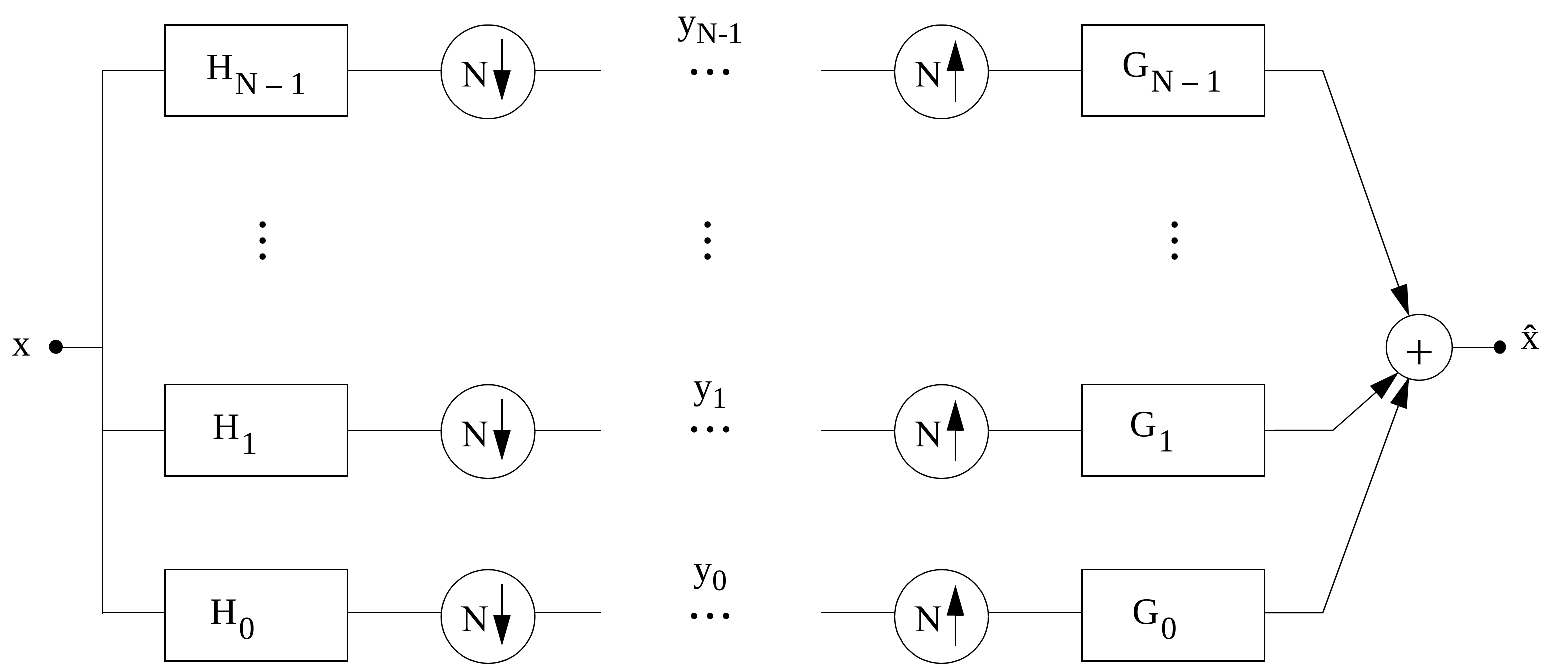




Multichannel filter banks

[Vaidyanathan & Doganata]: **lattice structures**

- similar to the two-channel case
- complete factorization

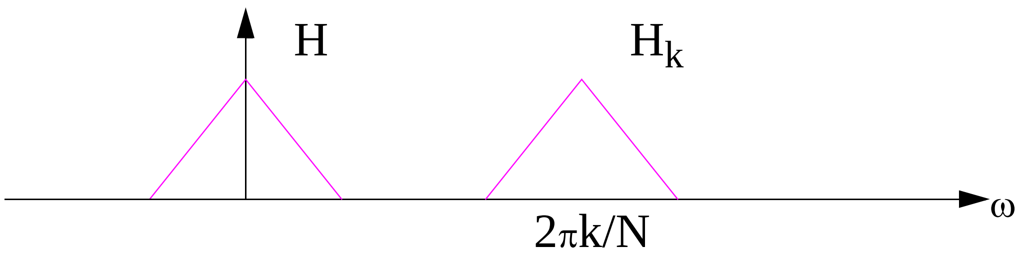


Modulated filter banks

All filters derived from one prototype by modulation

$$H_k(z) = H(W_N^k z)$$

$$h_k[n] = \left\{ h[0], W_N^{-k} h[1], W_N^{-2k} h[2], W_N^{-3k} h[3], \ldots \right\}$$

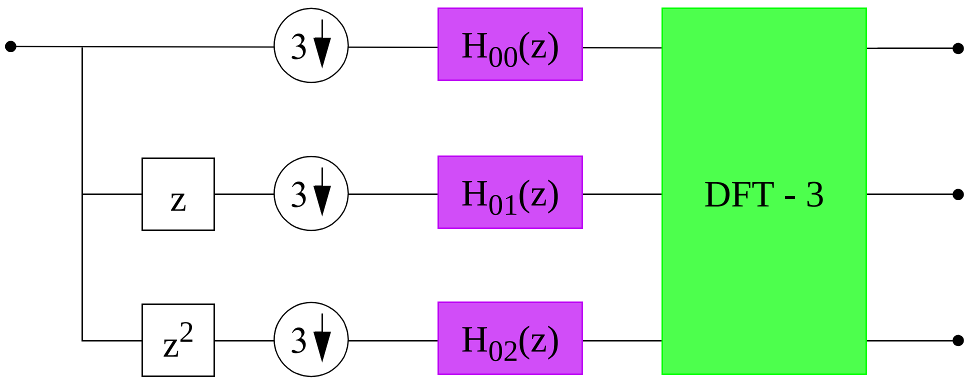


Polyphase matrix has a particular form (ex. N=3)

$$\begin{bmatrix} H_0 & H_1 & H_2 \\ H_0 & W^{-1}H_1 & W^{-2}H_2 \\ H_0 & W^{-2}H_1 & W^{-1}H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W^{-1} & W^{-2} \\ 1 & W^{-2} & W^{-1} \end{bmatrix} \cdot \begin{bmatrix} H_0 & 0 & 0 \\ 0 & H_1 & 0 \\ 0 & 0 & H_2 \end{bmatrix}$$

Modulated filter banks

Implementation: polyphase filters + DFT



Theorem: [Discrete Balian-Low]
There are no FIR perfect reconstruction modulated filter banks with modulation by the roots of unity and critical sampling. In particular, there are no orthonormal solutions.

Local cosine bases ...
... answer to time-frequency problem

How does one achieve locality in time and frequency?

- solution: divide the axis into intervals and construct Fourier series on them

Problems

- convergence slow and approximation poor
- discontinuity

How to divide the time axis?

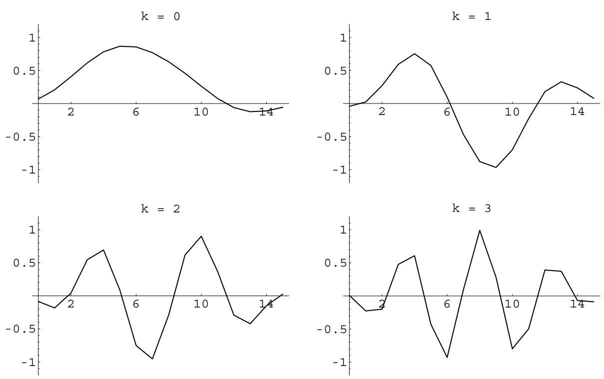
- improvement by local cosine bases
- discrete time: [Malvar, Princen, Bradley]
- continuous time: [Coifman & Meyer]
- smooth extension so that the functions overlap but with folding that guarantees orthogonality

Local cosine bases ...
... in discrete time

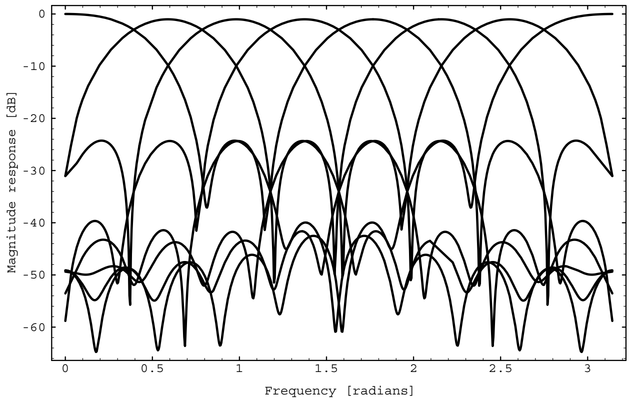
Filters: modulated prototype filter

$$h_k[n] = w[n] \times \cos\left(\frac{2\pi(2k+1)}{4N}\left(n - \frac{2N-1}{2}\right) + \frac{\pi}{4} + \frac{k\pi}{2}\right)$$

impulse responses

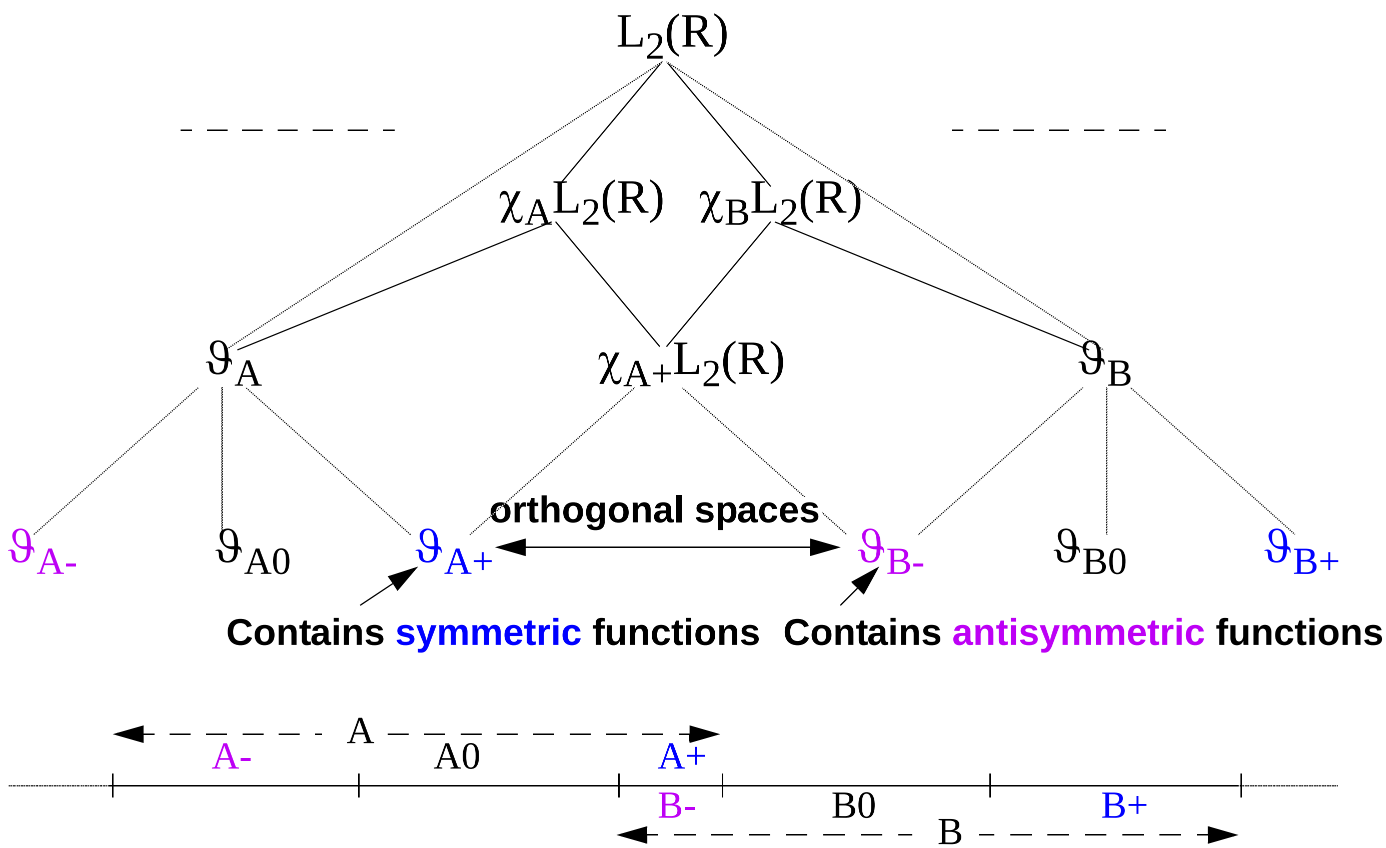


magnitude responses



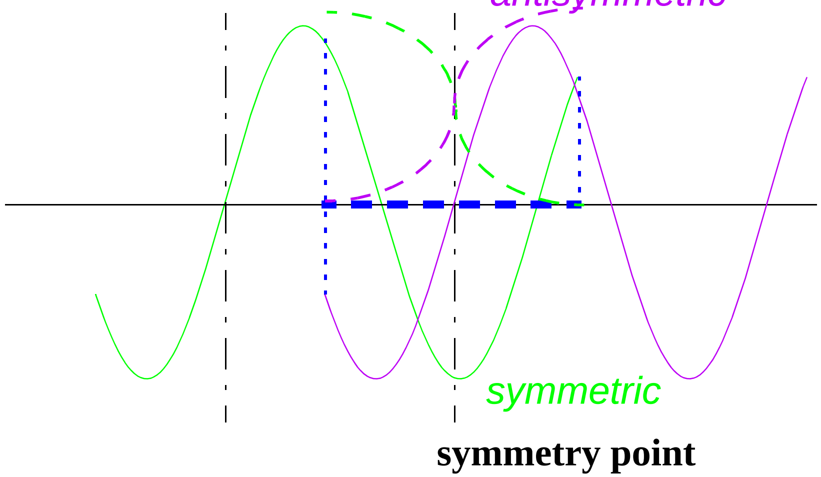
Local cosine bases ...
... why does it actually work?

Reformulate the problem [Bernardini & Kovacevic]

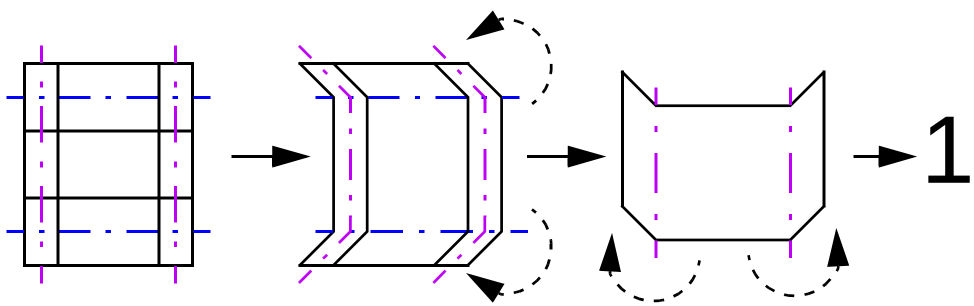


Local cosine bases

Orthogonality of two subspaces ϑ_{A+} and ϑ_{B-}



- use projections to build functions in subspaces
- window satisfies power-complementarity



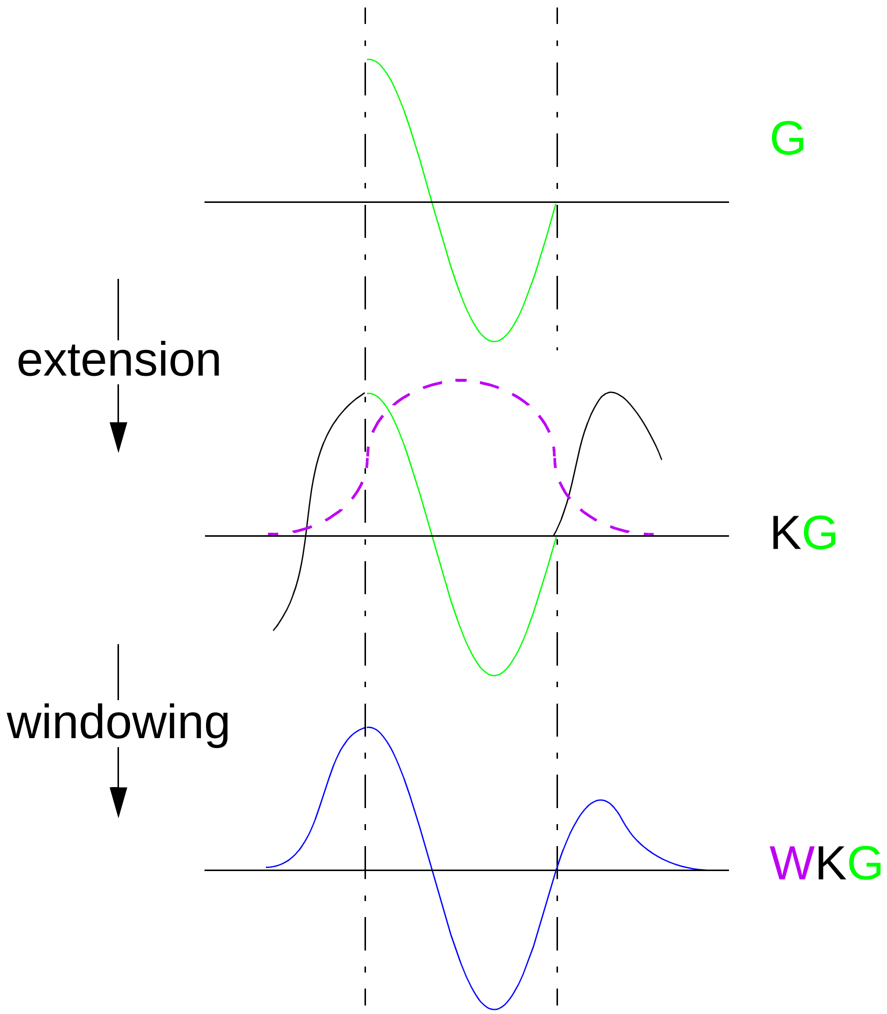
Local cosine bases ...
... building the bases

General structure of an orthonormal basis

$$H = WKG$$

Properties

- fast implementation
- one prototype filter
- can switch between different transforms
- any # of dimensions
- starting basis not necessarily cosine



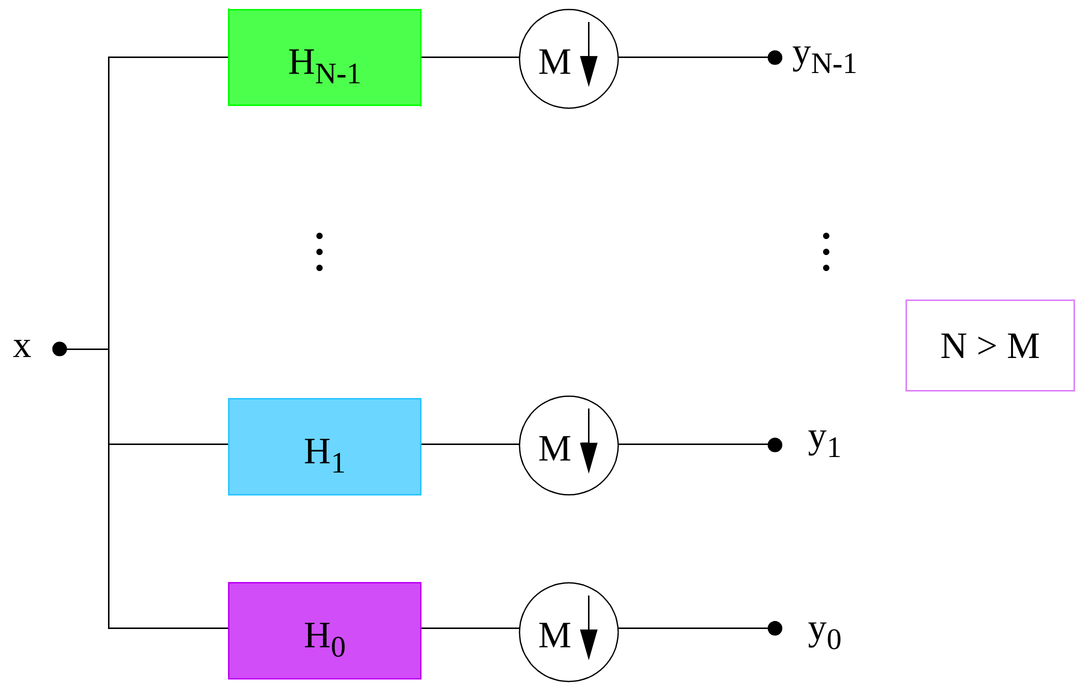
Local orthogonal bases ...
... summary

- one prototype filter
- can switch between different transforms
- any # of dimensions
- flexible design tool
- design bases with prescribed properties
- complexity low

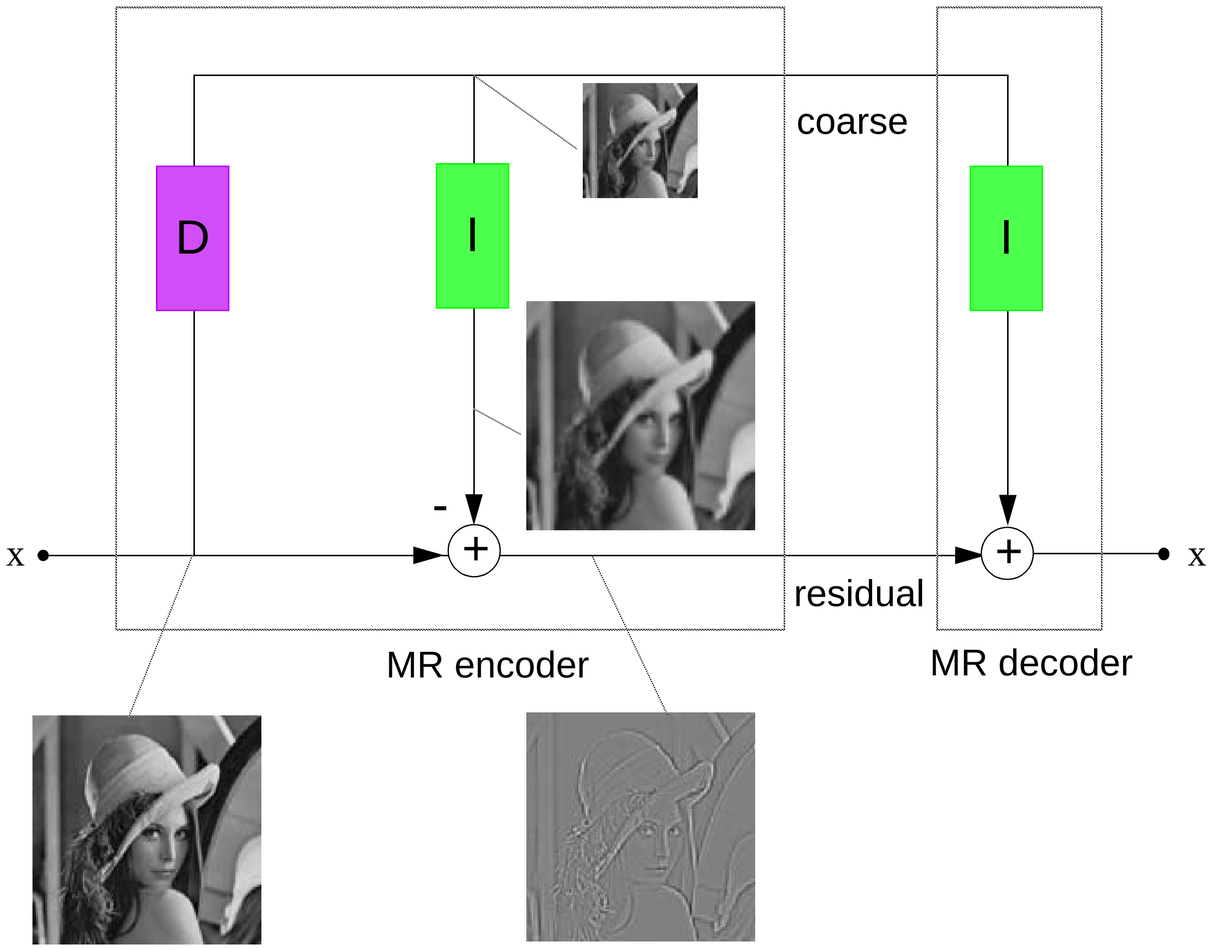
Previous		Local Orthogonal Bases		
μ	α		μ	α
256	248	**W**	32	/
		K	/	24
		G	64	56
256	248		96	80

- used in almost all standard audio coders

Oversampled filter banks

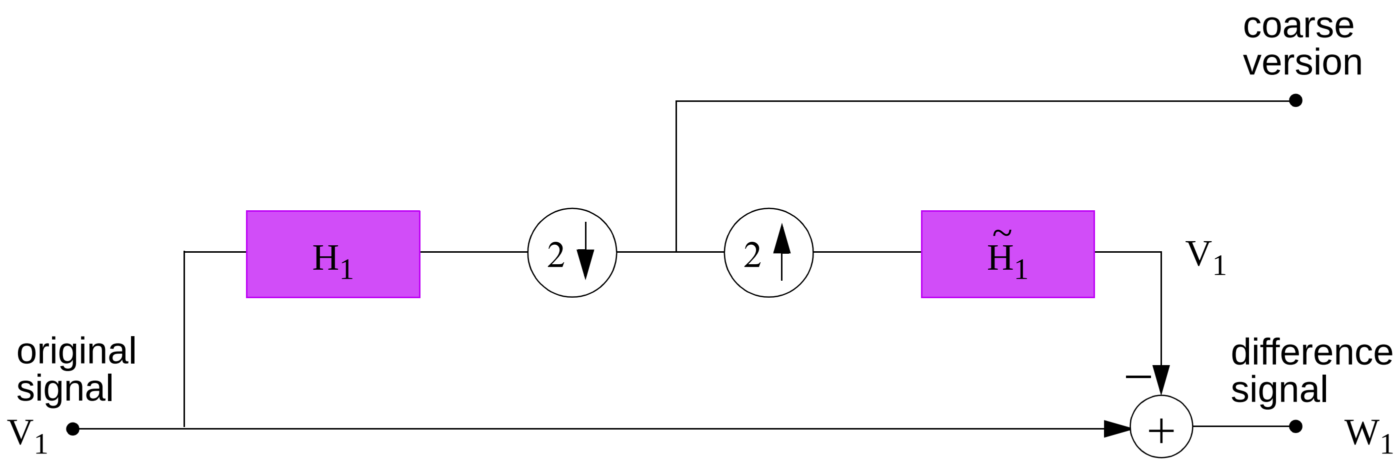


Pyramid scheme



Pyramid scheme

Orthogonal filter used



Advantages

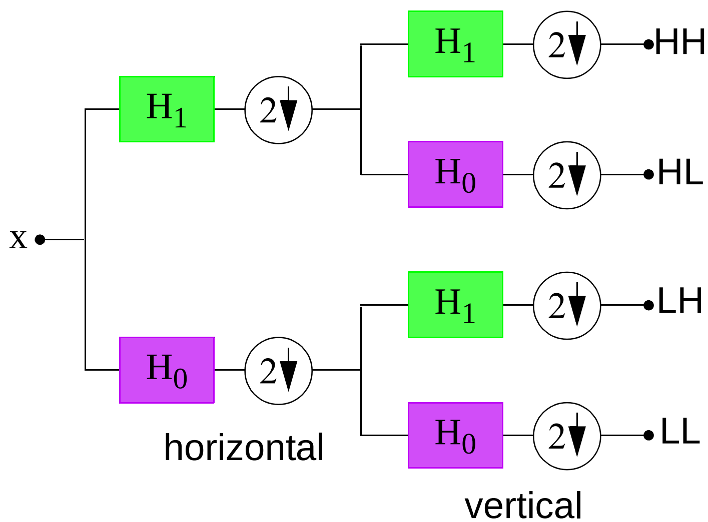
- robustness
- decimation and interpolation operators: arbitrary

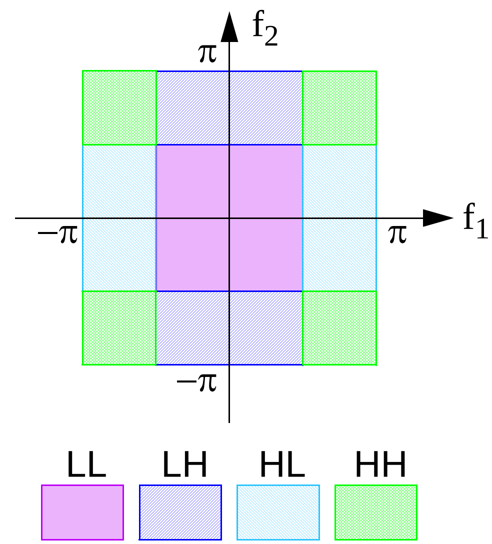
Drawbacks

- not critically sampled

Multidimensional filter banks

Main approach: use 1D techniques in a separable fashion





Problems

- very constrained filter design (separable filters)
- just rectangular spectrum divisions possible
- some solutions are not possible (ex. LP + ON)

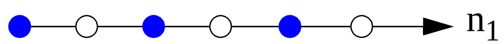
Problem with MD: lack of factorization theorems

Multidimensional filter banks ...
... sampling

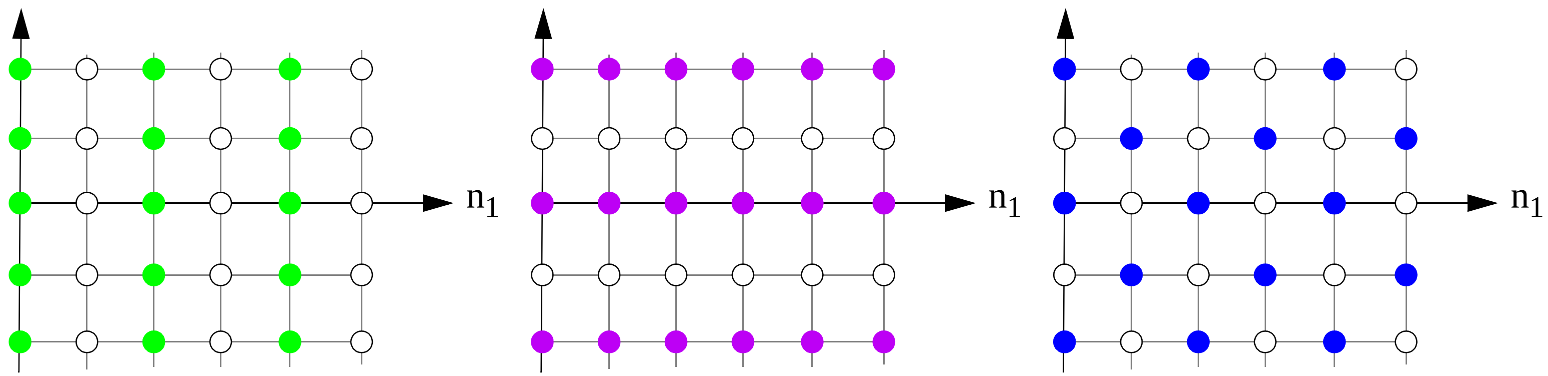
Main difference when compared to 1D:
sampling represented by a lattice

To sample by 2:

- in 1D

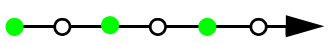


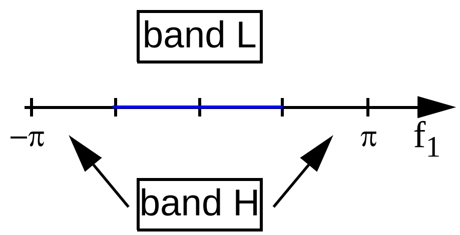
- in 2D



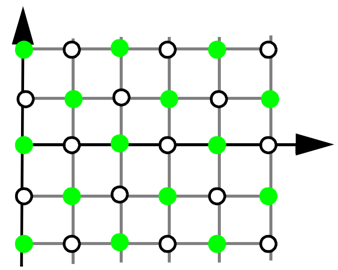
Multidimensional filter banks
... two-channels

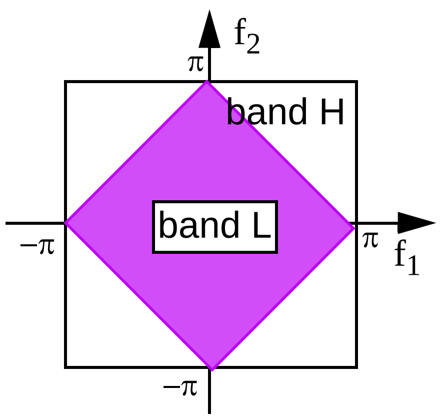
in 1D



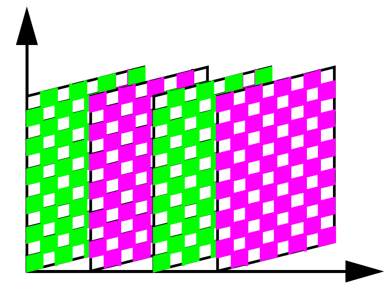


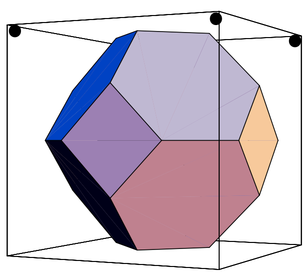
in 2D





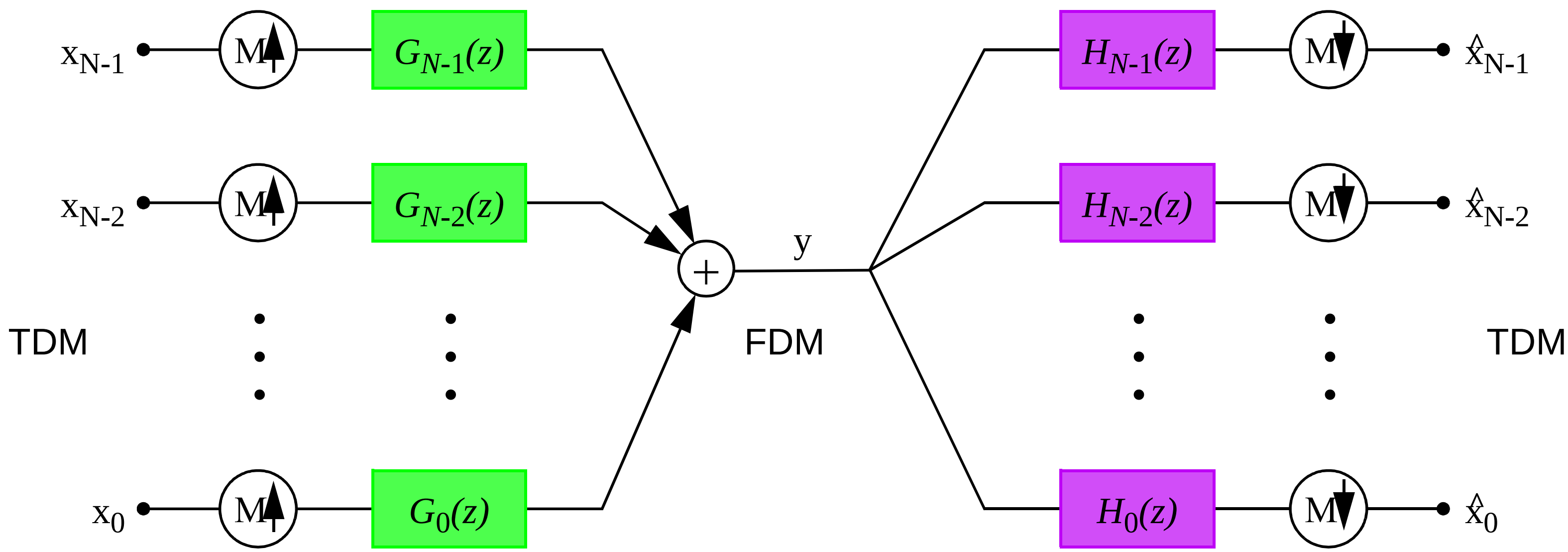
in 3D





Transmultiplexers

Dual to filter banks: used in telecommunications



Analysis similar to that of filter banks