Series Expansions Using Wavelets and Modulated Bases

“All this time, the guard was looking at her;
first through a telescope, then through a microscope,
and then through and opera glass.”

Lewis Carroll, *Through the Looking Glass*

Definition of the problem

Multiresolution concept and analysis

Construction of wavelets using Fourier techniques

Wavelets derived from iterated filter banks and regularity

Wavelet series and its properties

Generalizations in one dimension

Multidimensional wavelets

Local cosine bases

Wavelets ...
... what are they and how to build them?

Orthonormal bases of wavelets

- Haar's construction of a basis for $L_2(\Re)$ (1910)
- Meyer, Battle-Lemarié, Stromberg (1980's)
- Mallat and Meyer's multiresolution analysis (1986)

Wavelets from iterated filter banks

- Daubechies' construction of compactly supported wavelets
- smooth wavelet bases for $L_2(\Re)$ and computational algorithms

Relation to other constructions

- successive refinements in graphics and interpolation
- multiresolution in computer vision
- multigrid methods in numerical analysis
- subband coding in speech and image processing

Goal: $\psi(t)$ **such that its scale and shifts form an orthonormal basis for** $L_2(\Re)$**.**

Wavelets ...
... definition of the problem

- series expansions for $L_2(\Re)$
- good time and frequency localization
- orthonormal bases and expansion: $\langle \varphi_i, \varphi_j \rangle = \delta[i-j]$

$$f = \sum_i \langle \varphi_i, f \rangle \varphi_i$$

- piecewise Fourier series:
  poor frequency localization, Gibbs
- local cosine bases

Wavelet series: given a wavelet $\psi(t)$, then the following

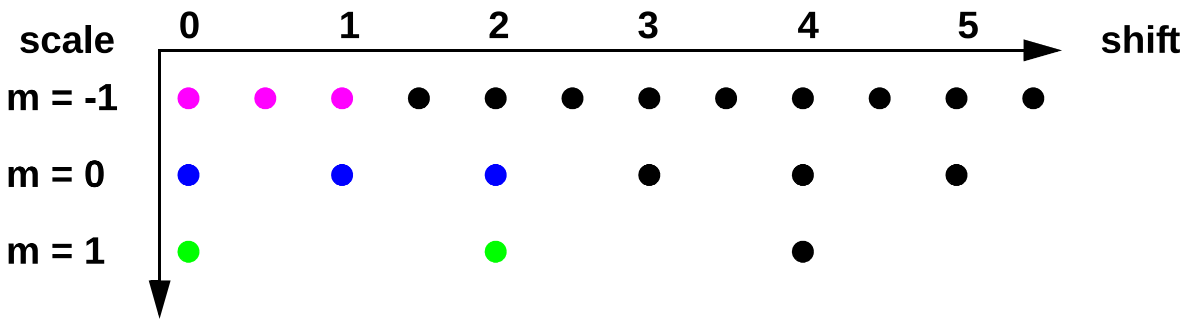
$$\psi_{mn}(t) = 2^{-m/2} \cdot \psi(2^{-m} t - n) \qquad m, n \in \Im$$

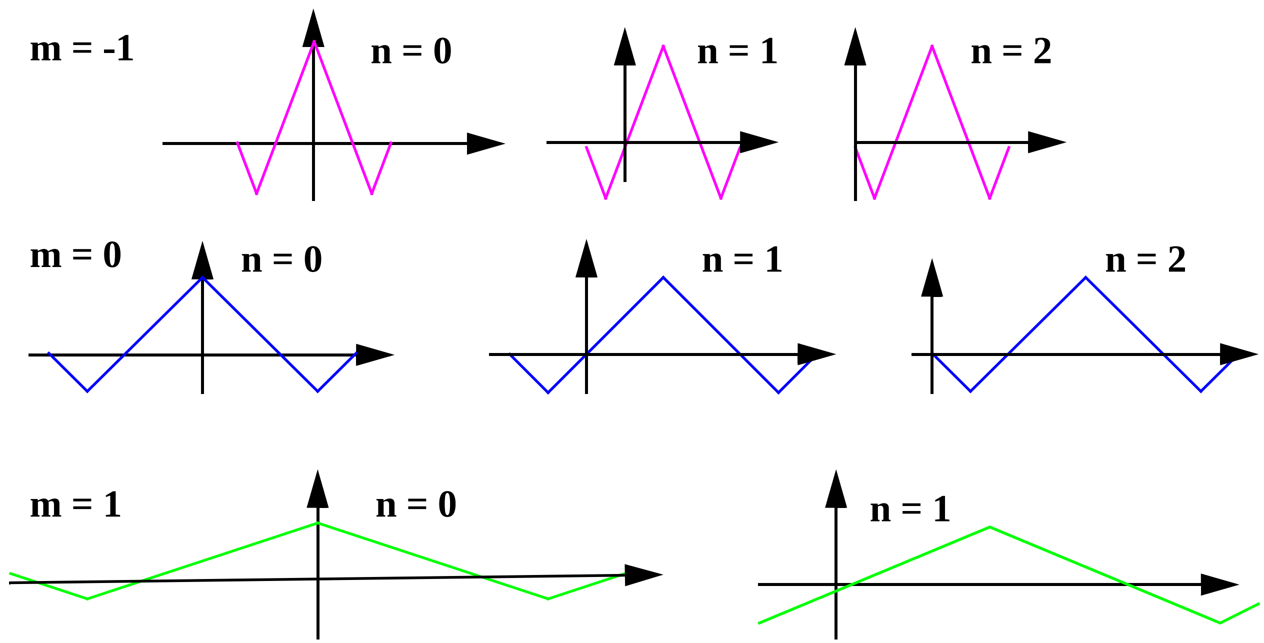
is an orthonormal basis for $L_2(\Re)$: $f(t) = \sum_m \sum_n \langle \Psi_{mn}, f \rangle \Psi_{mn}(t)$

- famous example: Haar in 1910

Wavelets

$$\psi_{mn}(t) = 2^{-m/2} \cdot \psi(2^{-m}t - n) \qquad m, n \in \Im$$



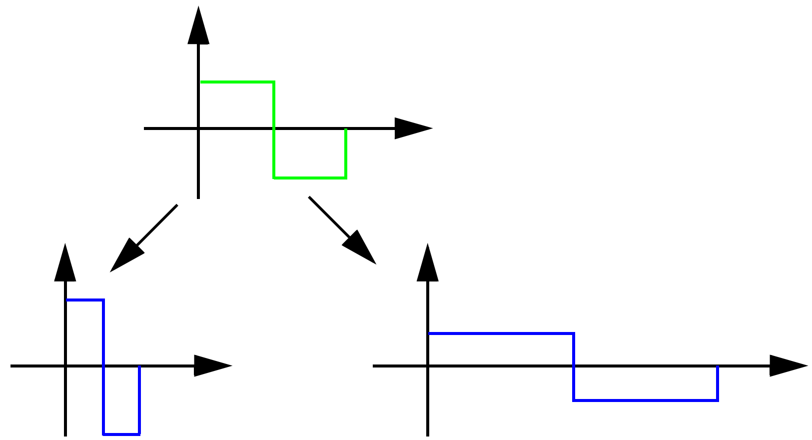


Haar system

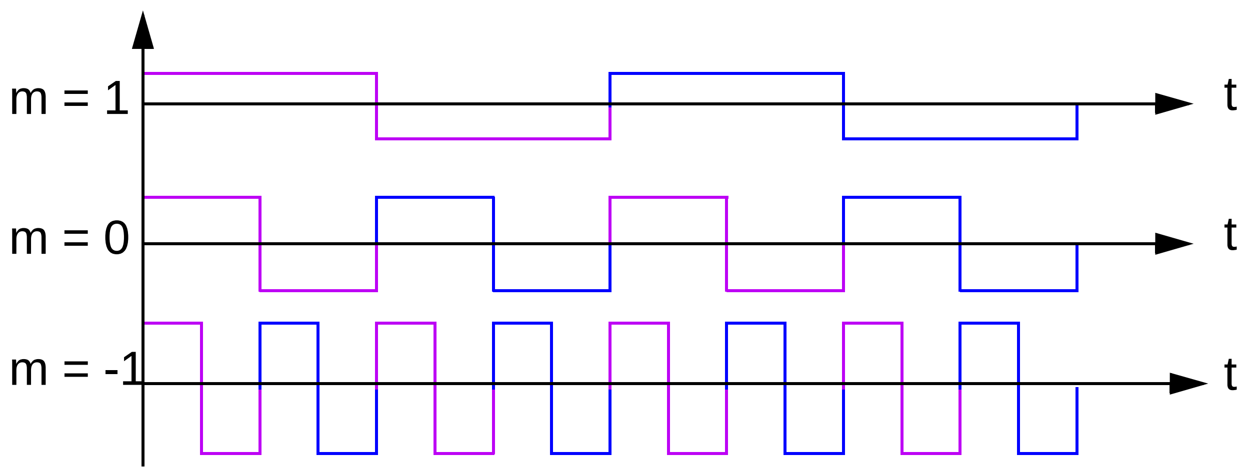
Basis functions

$$\psi(t) = \begin{cases} 1 & 0 \le t < 0.5 \\ -1 & 0.5 \le t < 1 \\ 0 & \text{else} \end{cases}$$

$$\psi_{mn}(t) = 2^{-m/2}\psi(2^{-m}t - n)$$



Basis functions across scales



Haar system...
... scaling function and wavelet

The Haar scaling function (indicator of unit interval)

$$\varphi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{else} \end{cases}$$

helps in the construction of the wavelet, since

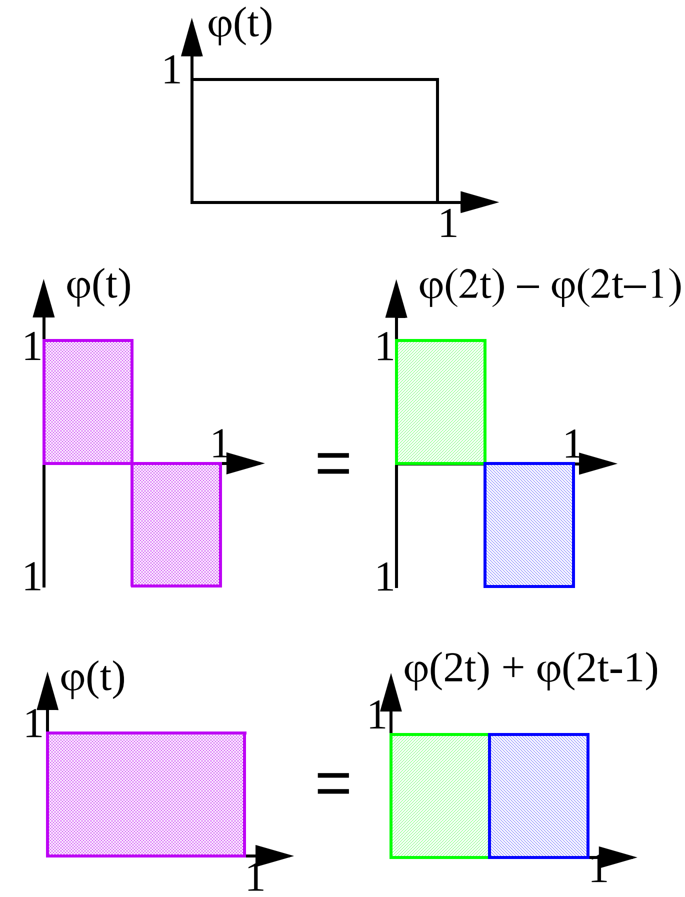
$$\psi(t) = \varphi(2t) - \varphi(2t-1)$$

and satisfies a two-scale equation

$$\varphi(t) = \varphi(2t) + \varphi(2t-1)$$

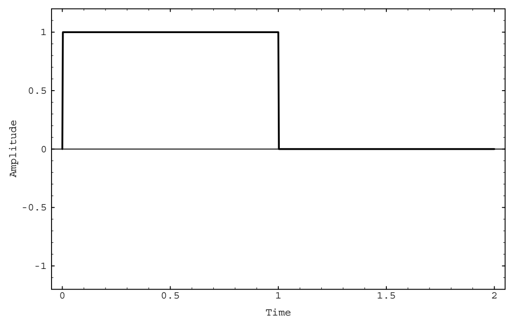
Note:

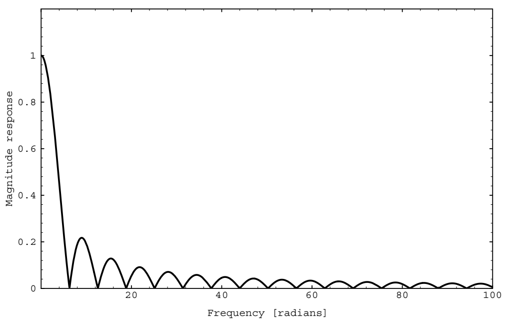
Haar wavelet a bit too trivial to be useful...



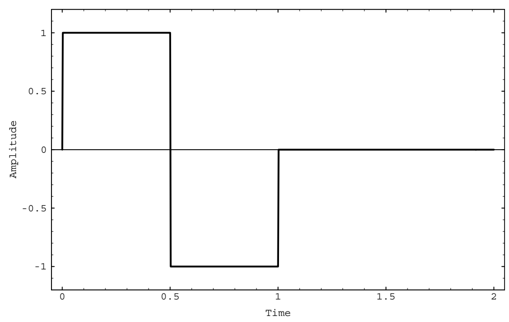
Haar system...
... scaling function and wavelet

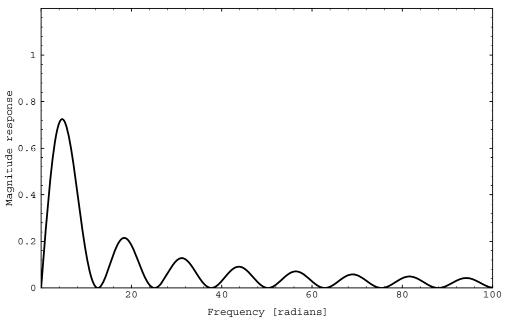
scaling function





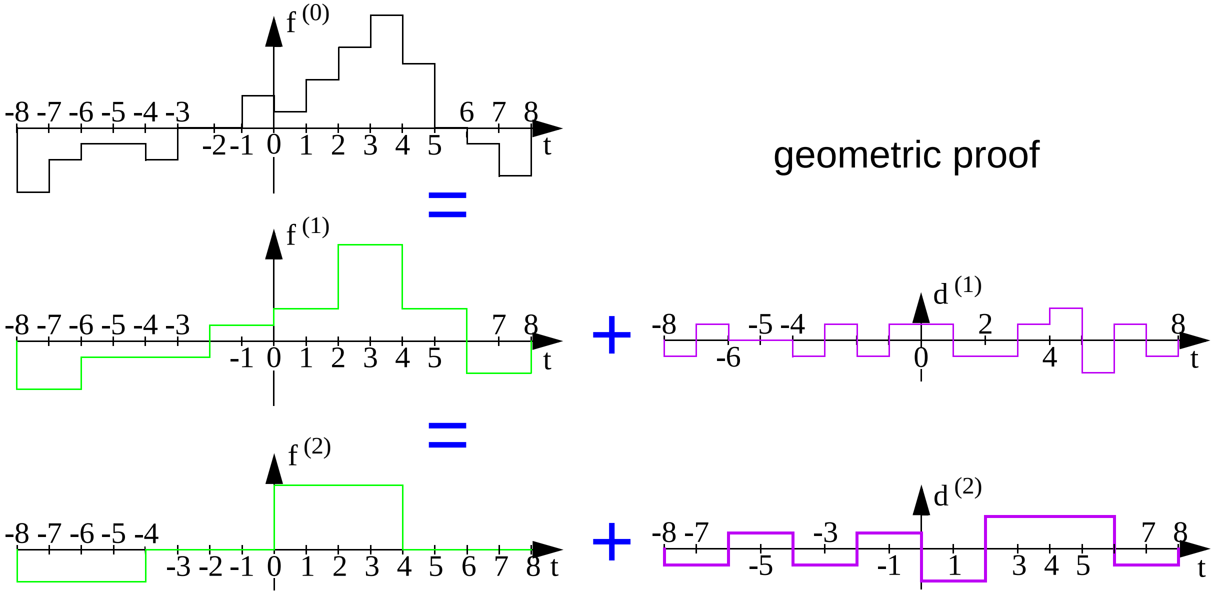
wavelet





Haar system...
... proof it is an orthonormal basis for $L_2(\Re)$

- piecewise constant functions are dense in $L_2(\Re)$
- write $f^{(i)} = f^{(i+1)} + d^{(i+1)}$
- show that $|f^{(i)}| \to 0$ as $i \to \infty$



geometric proof

Note: two functions are involved

- scaling function $\varphi^{(i)}(t)$, to go from $f^{(i-1)}$ to $f^{(i)}$
- wavelet $\psi^{(i)}(t)$ to represent the difference $d^{(i)}$

Proof that Haar is an orthonormal basis for $L_2(\Re)$

A. Piecewise constant functions are dense in $L_2(\Re)$

- pieces of arbitrarily small size, 2^i, $i \to -\infty$
- $|f - f^{(i)}|_2 \to 0$ as $i \to -\infty$

B. Write function as a sum and a difference

- $f^{(i)} = f^{(i+1)} + d^{(i+1)}$
- this can be done on coefficients of the expansion
- $f^{(i+1)}$ is a linear combination of scaling functions of size 2^i
- $d^{(i+1)}$ is a linear combination of wavelets of size 2^i

C. Iterate

- $f^{(i)} = f^{(i+N)} + \sum_{m=1}^{N} d^{(i+m)}$
- $f^{(i+N)}$ is a linear combination of scaling functions of size 2^N
- $d^{(i+m)}$ is a linear combination of wavelets of size 2^m, $m = 1, \ldots, N$

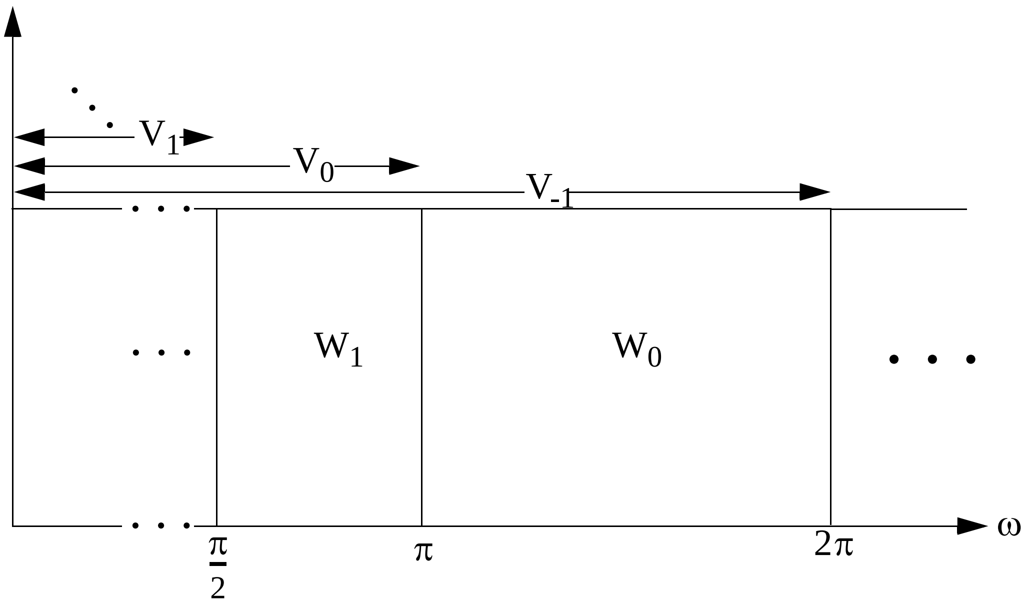
D. Limit as $N \to \infty$

- show that $|f^{(N)}|_2 \to 0$ as $N \to \infty$
- thus, $|f^{(i)} - \sum d^{(i+m)}|_2 \to 0$ with $m = 1, \ldots, N$, as $N \to \infty$

Sinc system ...
... bandlimited function spaces

Cover $L_2(\Re)$ with octave bands

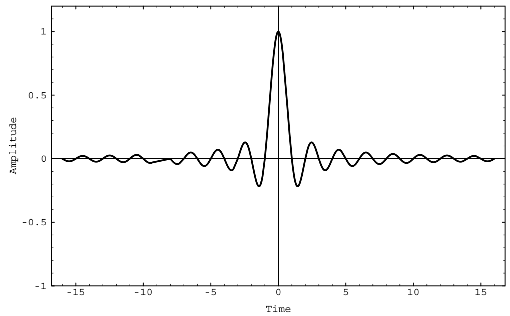
- V_i: $[-2^{-i}\pi, 2^{-i}\pi]$
- W_i: $[-2^{-i+1}\pi, -2^{-i}\pi]$ & $[2^{-i}\pi, 2^{-i+1}\pi]$

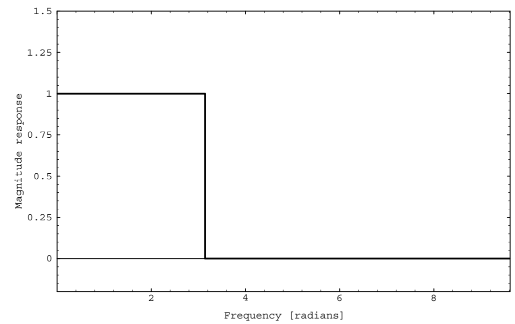


- orthonormal basis for V_i : sinc functions
- orthonormal basis for W_i : difference of sinc functions
- successive approximation by more and more octave bands

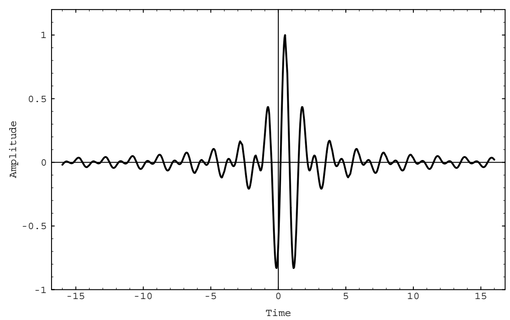
Sync system ...
... scaling function and wavelet

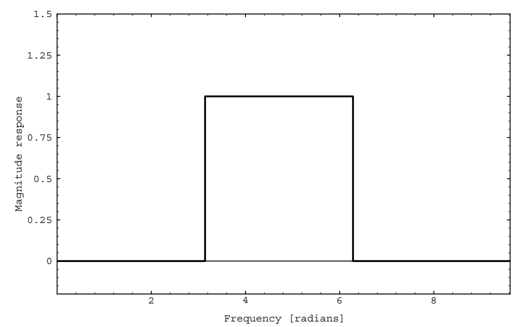
scaling function





wavelet

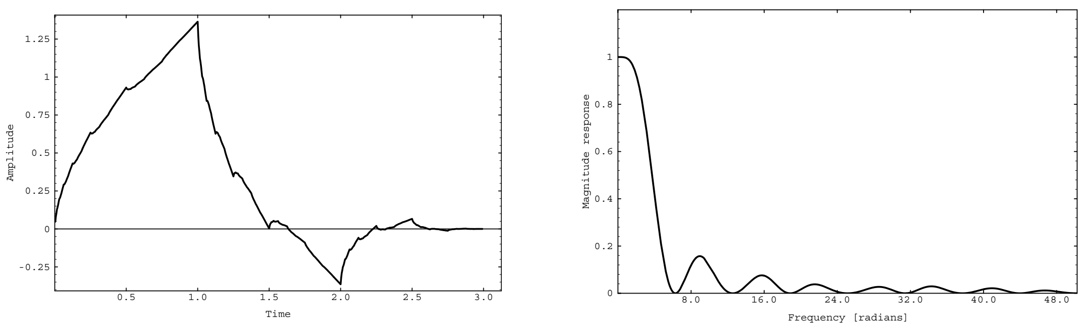




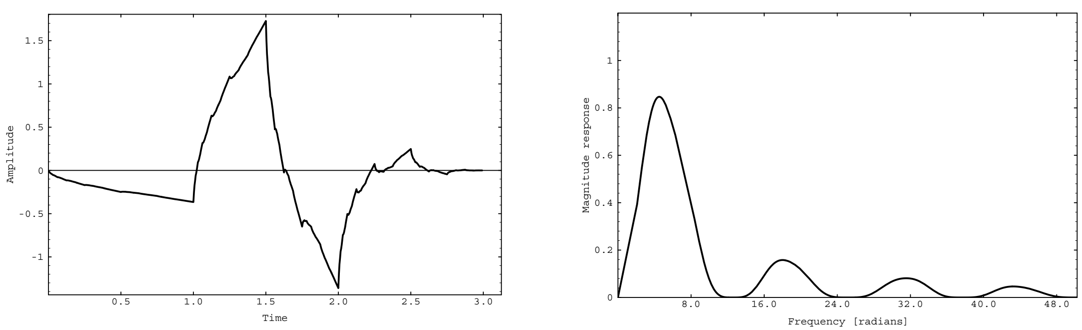
Daubechies' construction...
... scaling function and wavelet

- Haar and sync systems: either good time OR frequency localization
- Daubechies system: good time AND frequency localization

scaling function



wavelet



Finite length, continuous $\varphi(t)$ and $\psi(t)$, based on L=4 **iterated filter**

Multiresolution concept and analysis

Multiresolution analysis for $L_2(\Re)$ [Mallat, Meyer]

- ladder of spaces

$$V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2}$$

- completeness

$$\bigcup_i V_i = L_2(\Re), \quad \bigcap_i V_i = \varnothing$$

- scaling property

$$f(t) \in V_i \Leftrightarrow \quad f(2^i t) \in V_0$$

- shifting property

$$f(t) \in V_0 \Leftrightarrow \quad f(t-n) \in V_0$$

- existence of an orthonormal basis for V_0

$$\{\varphi(t-n)\} \qquad n \in \Im$$

Multiresolution concept and analysis

Assume a basis for V_0 **given by** $\{\varphi(t-n)\},\ n \in \Im$

- since $\varphi(2t-n)$ is a basis for V_{-1} (use $V_0 \subset V_{-1}$)

$$\varphi(t) = \sqrt{2}\sum g_0[k] \cdot \varphi(2t-k)$$

- orthogonal complement to V_0 in V_{-1}

$$V_{-1} = V_0 \oplus W_0$$

- basis for W_0: wavelet

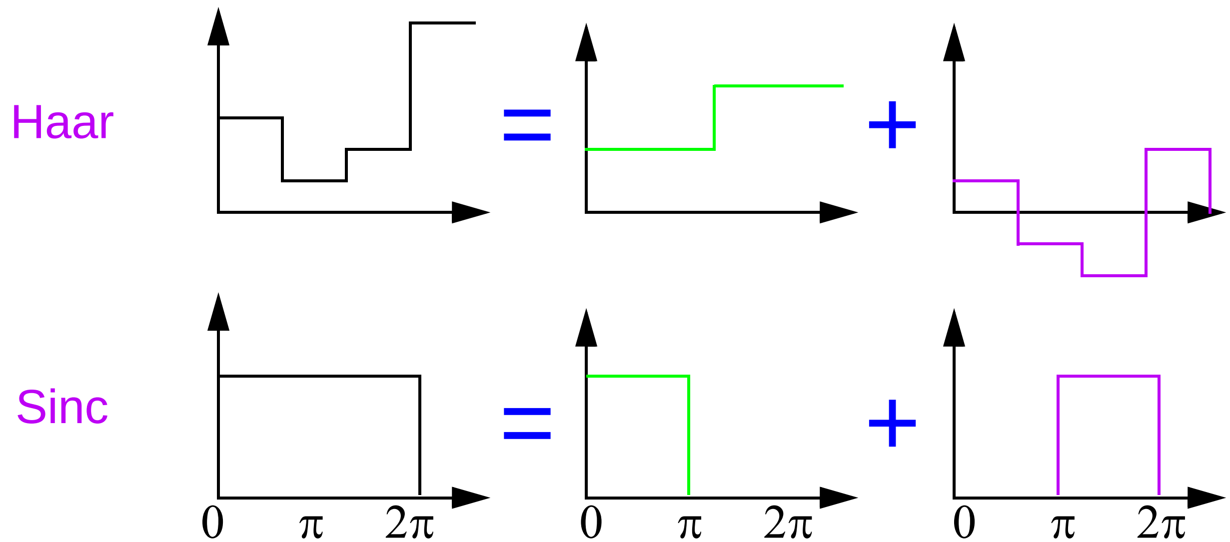
$$\psi(t) = \sqrt{2}\sum g_1[k] \cdot \varphi(2t-k)$$

$$g_1[n] = (-1)^n \cdot g_0[-n+L-1]$$

Multiresolution concept and analysis

Basic examples

- Sinc $\qquad V_0 = BL[0, \pi],\ W_0 = BL[\pi, 2\pi]$
- Haar $\qquad V_0 = \text{const}[n, n+1],\ W_0 = \text{difference}$
- Daubechies $\qquad$ between Haar and sinc



Multiresolution analysis ...
... iteration

Decomposition

$$V_{-1} = W_0 \oplus V_0$$

- basis for V_0: $\varphi(t-n)$
- basis for W_0: $\psi(t-n)$

Iterate the decomposition

$$V_{-1} = W_0 \oplus W_1 \oplus W_2 \oplus W_3$$

Consider V_N as $N \to -\infty$, and the above decomposition

Limit

$$L_2(\Re) = \oplus W_i$$

Construction of bases ...
... Fourier method

Method of Meyer and Lemarié

Idea: construct a scaling function that:

- satisfies a two-scale equation

$$\varphi(t) = \sqrt{2}\sum g_0[k] \cdot \varphi(2t-k) \quad \Leftrightarrow \quad \Phi(\omega) = \frac{1}{\sqrt{2}} G_0(e^{j\omega/2})\Phi(\omega/2)$$

- is orthogonal to its integer translates

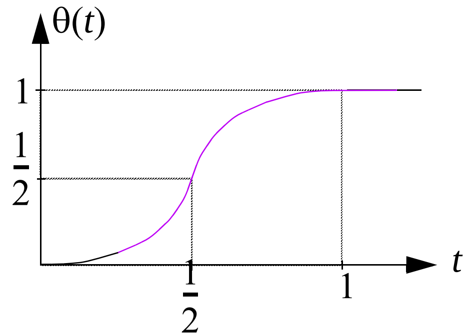
$$\langle \varphi(t), \varphi(t-n) \rangle = \delta[n] \quad \Leftrightarrow \quad \sum_k |\Phi(\omega + k2\pi)|^2 = 1$$

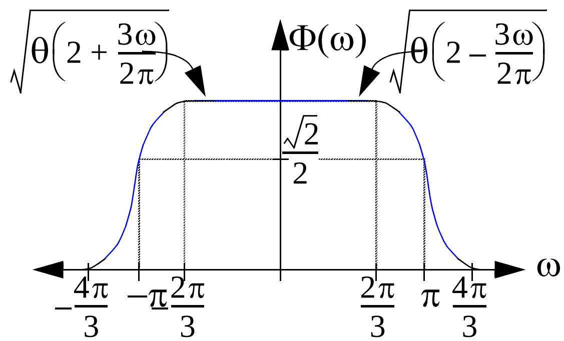
- then, using the multiresolution framework,
  we will get the wavelet

Construction of bases...
... Fourier method

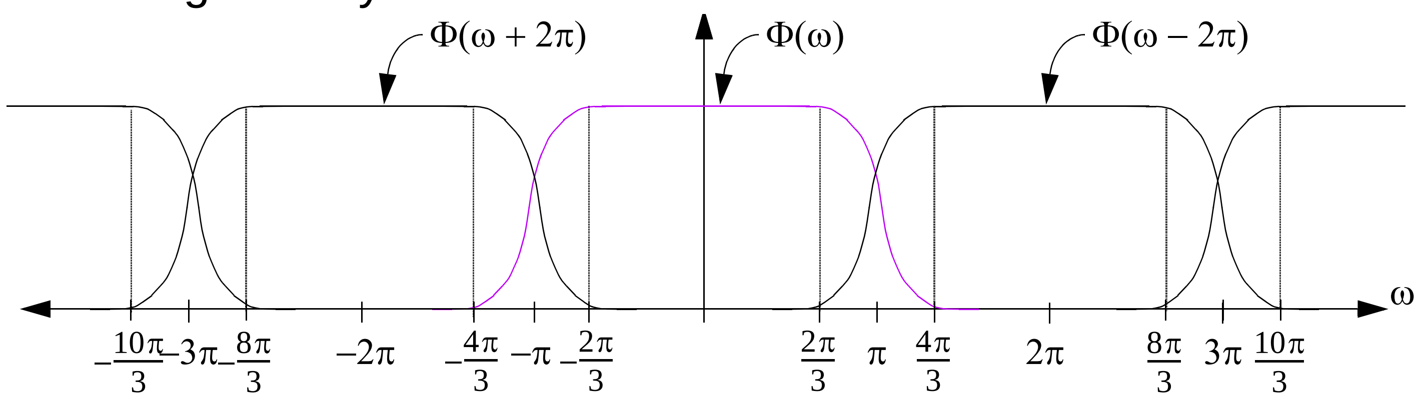
Inspiration: sinc function

- use as transition function $\theta(t)$ that smooths sinc
- $\theta(t)$ satisfies $\theta(t) + \theta(1-t) = 1$





- orthogonality



$$\sum_k |\Phi(\omega + k2\pi)|^2 = 1$$

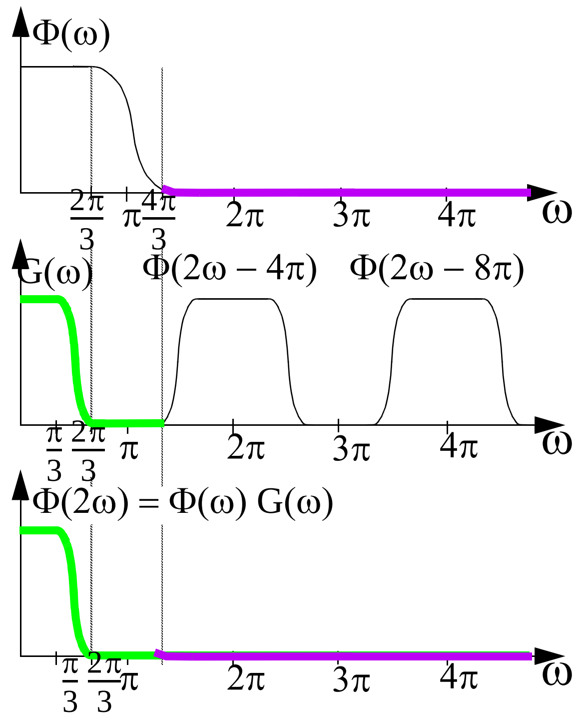
Construction of bases...
... Fourier method

- two-scale equation

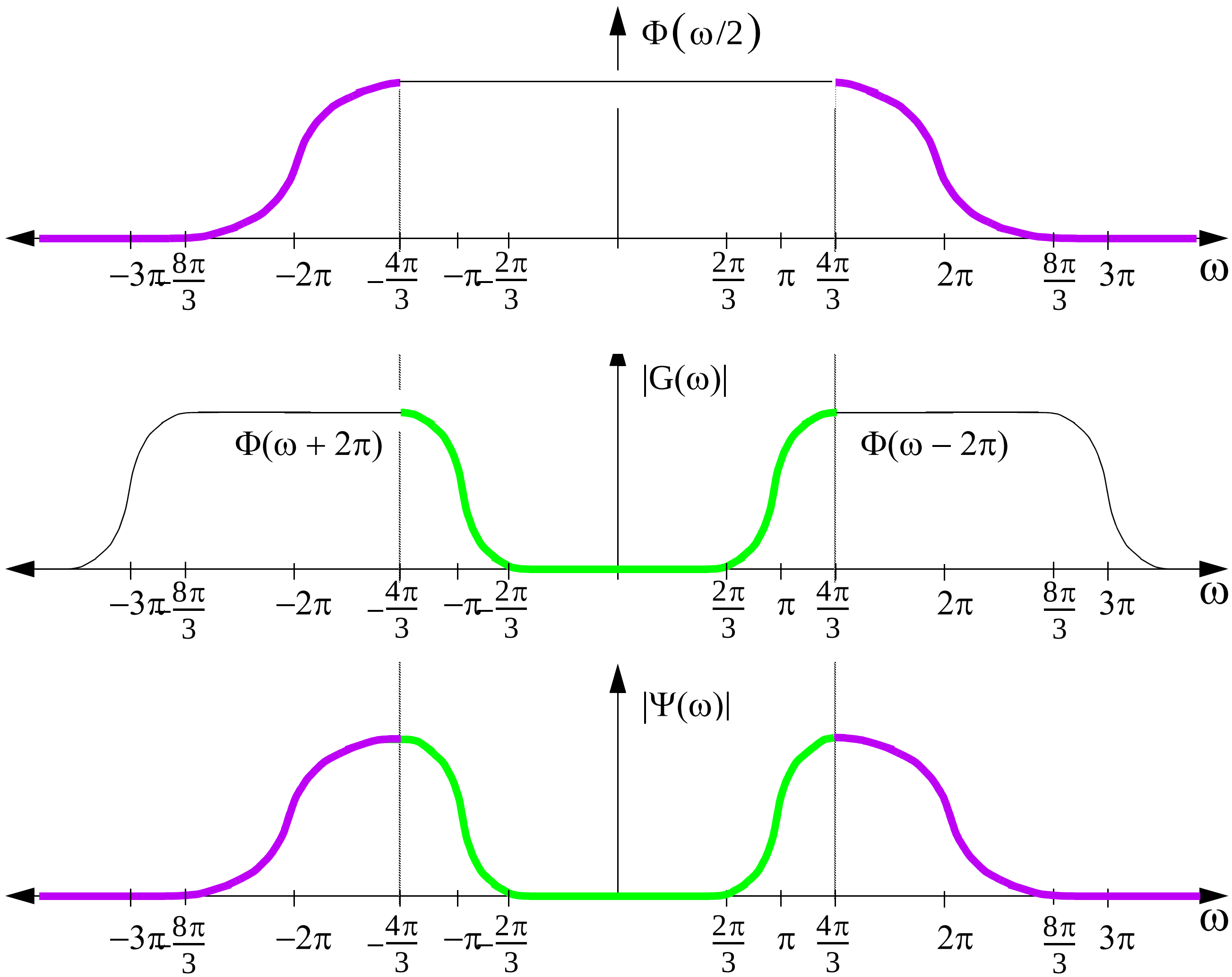
$$\varphi(t) = \sqrt{2}\sum g_0[n]\varphi(2t-n)$$

$$\Phi(\omega) = \frac{1}{\sqrt{2}}G_0(e^{j\omega/2})\Phi(\omega/2)$$

$$\Phi(2\omega) = \frac{1}{\sqrt{2}}G_0(e^{j\omega})\Phi(\omega)$$

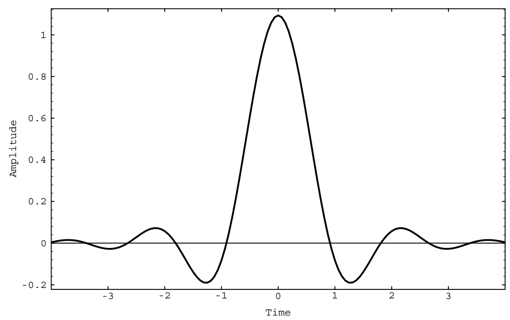


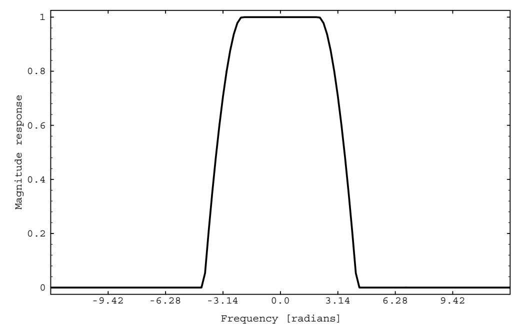
Meyer's system



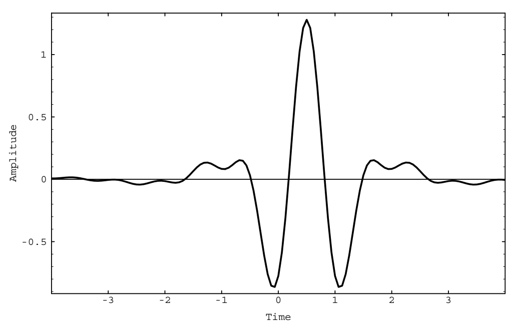
Meyer's system ...
... scaling function and wavelet

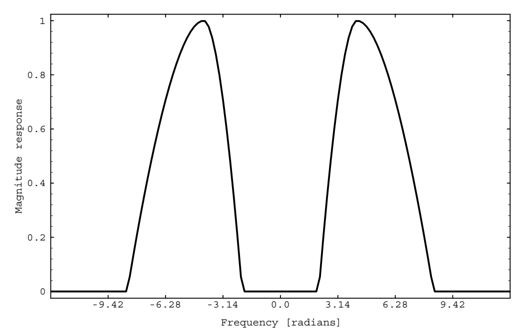
scaling function





wavelet



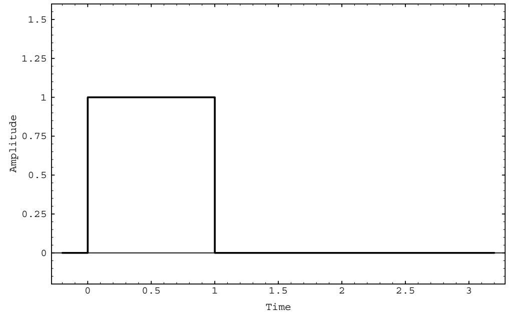


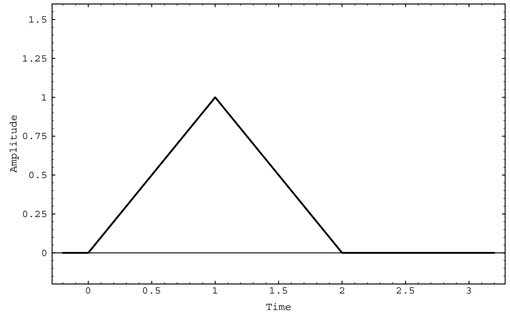
Splines

Piecewise polynomial spaces

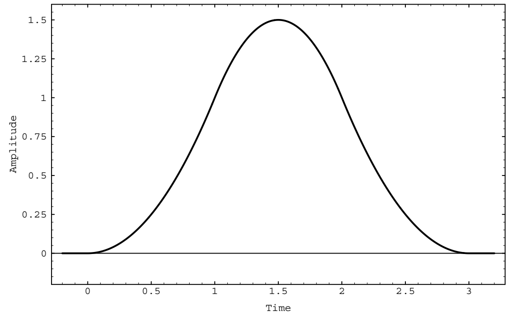
- bases are B-spline
  convolution of box function with itself
- they are not orthogonal, need to orthogonalize
- Battle-Lemarié

constant





linear



quadratic

Fourier method for splines

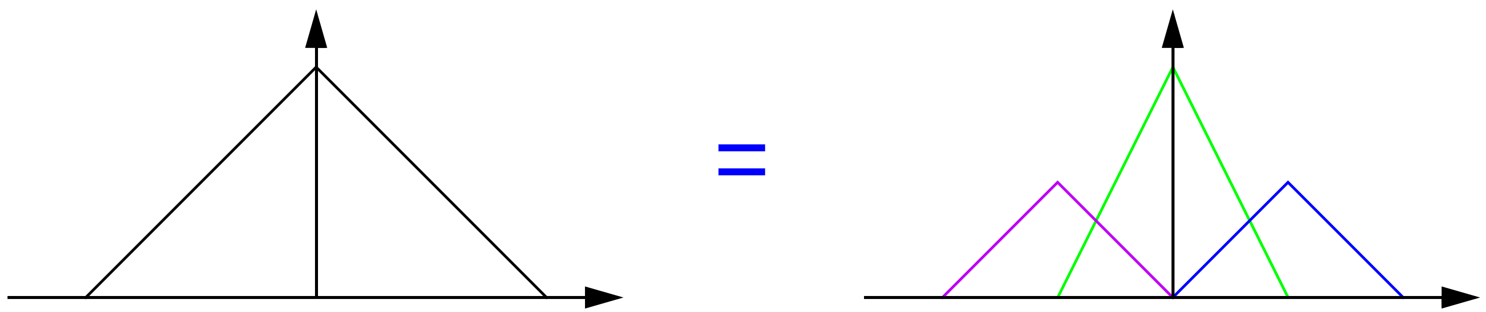
Consider linear spline spaces:

- V_0 = piecewise linear and continuous over $[n,n+1]$
- V_i = piecewise linear and continuous over $[2^i n, 2^i(n+1)]$

These spaces are embedded:

$$V_i \subset V_j \qquad i > j$$

We have a nonorthogonal basis for V_0**, namely the hat function** $h(t)$**, which satisfies a two-scale equation:**



$$h(t) = \frac{1}{2}h(2t-1) + h(2t) + \frac{1}{2}h(2t+1)$$

Fourier method for splines

Except orthogonality, all the axioms for multiresolution are satisfied + orthogonalize!

Recall $\langle \varphi(t), \varphi(t-n)\rangle = \delta[n] \Leftrightarrow \sum_k |\Phi(\omega + k2\pi)|^2 = 1$

Guess

$$\Phi(\omega) = \frac{H(\omega)}{\left(\sum_k |H(\omega + k2\pi)|^2\right)^{1/2}}$$

Then, the numerator is 2π periodic and one can verify

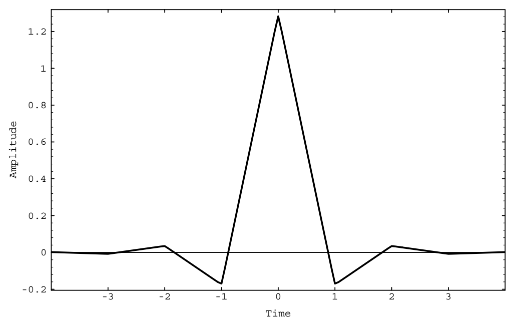
$$\sum_k |\Phi(\omega + k2\pi)|^2 = 1$$

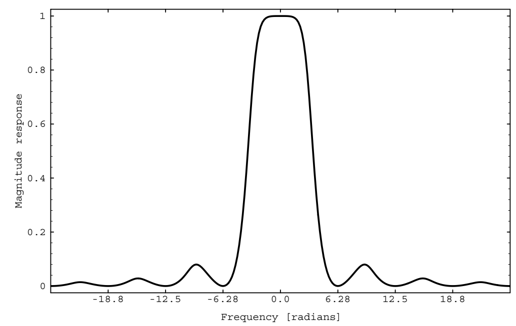
One can show that $\sum_k |H(\omega + k2\pi)|^2 > 0$ and thus the above is well defined

Now we have an orthogonal scaling function for V_0 and we find the wavelet using MR

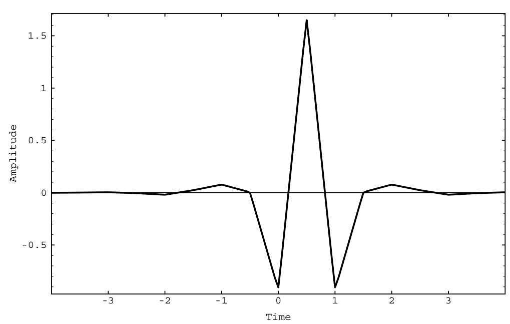
Splines ...
... a linear basis

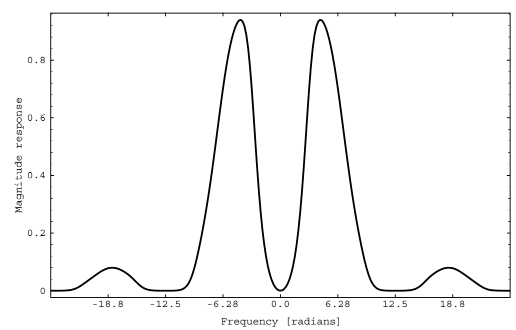
scaling function





wavelet





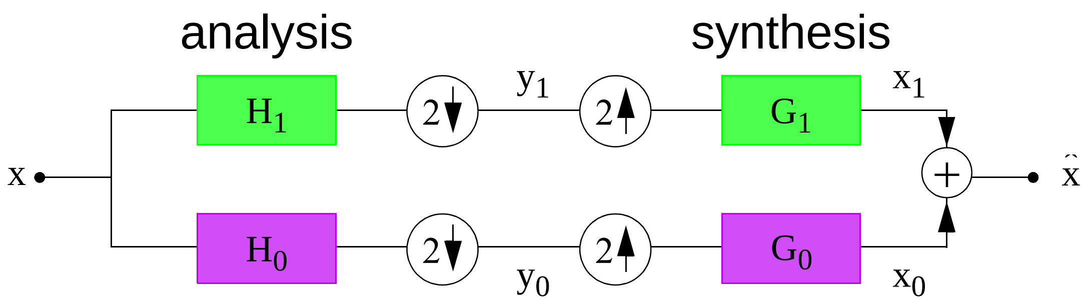
Splines ...
... 10 reasons to use them [Unser]

- closed-form representation
- simple manipulation
- symmetry
- shortest scaling function of order N
- maximum regularity for a given order N
- m-scale relation
- variational properties
- best approximation
- optimal time-frequency localization
- convergence to the ideal filter

Daubechies' system...
... iterated filter banks

Start with an orthonormal basis for $l_2(\mathfrak{I})$

- perfect reconstruction filter bank, FIR length L
- $\{g_0[n-2k], g_1[n-2k]\}$ is an orthonormal set ($h_i[n] = g_i[-n]$)



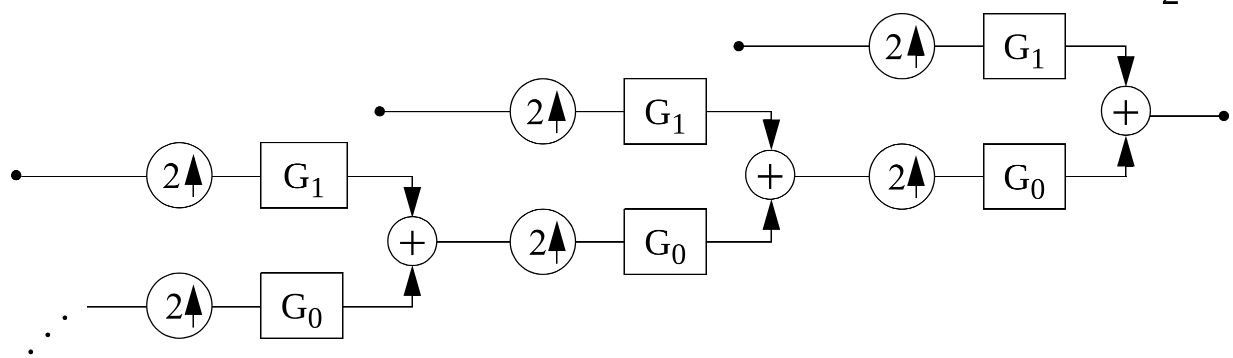
Consider $G_0(z) = \sum g_0[n] z^{-n}$

Orthonormality: $P(z) = G_0(z)G_0(z^{-1})$ **(deterministic ACF)**

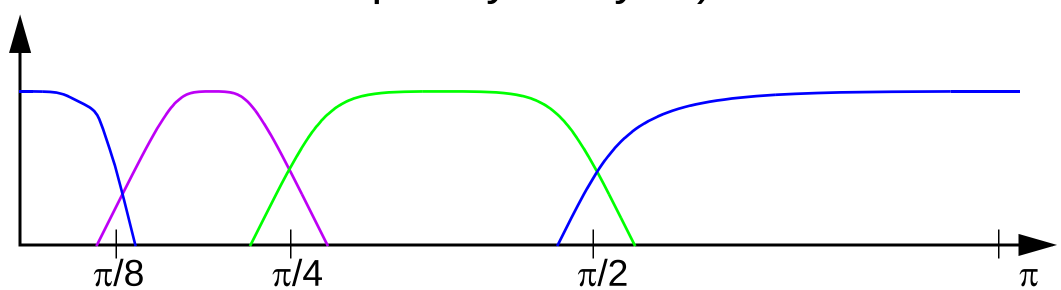
$$P(z) + P(-z) = 2$$

Daubechies' construction...
... iterated filter banks

Iteration will generate an orthonormal basis for $l_2(\mathfrak{I})$

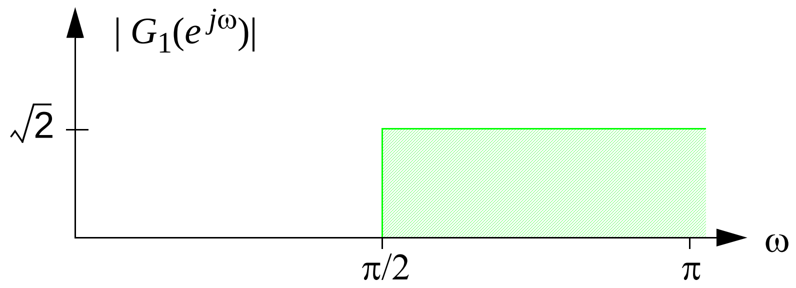


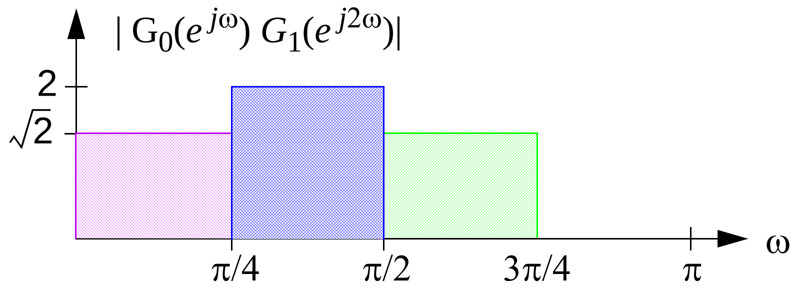
Consider equivalent basis sequences $G_0^{(i)}(z)$ **and** $G_1^{(i)}(z)$
(generates octave-band frequency analysis)

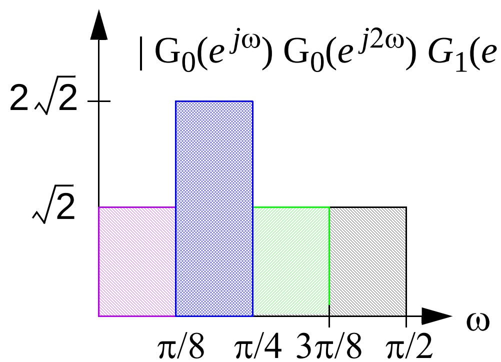


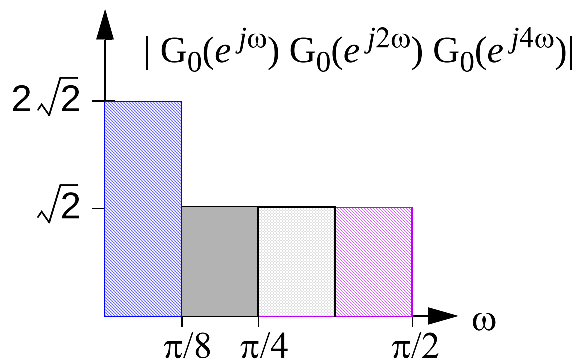
Interesting question: what happens in the limit?

Iterated filter banks...
... example









Daubechies' system...
... iterated filter banks

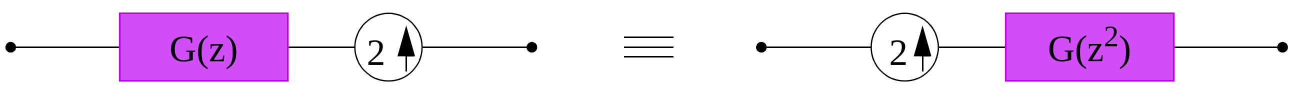
Consider equivalent LP basis sequences $G_0^{(i)}(z)$

$$G_0^{(i)}(z) = \prod_{k=0}^{i-1} G_0\left(z^{2^k}\right)$$

and bandpass $G_1^{(i)}(z)$

$$G_1^{(i)}(z) = G_1\left(z^{2^{i-1}}\right) \prod_{k=0}^{i-1} G_0\left(z^{2^k}\right)$$

Proof:



Thus: applying the above identity i times leads to

$$G(z) \cdot G(z^2) \cdot G(z^4) \cdot \ldots \cdot G(z^{i-1})$$

Daubechies' system...
... iterated filter banks

Associate piecewise constant approximation

$$\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[k], \qquad \frac{k}{2^i} < t < \frac{k+1}{2^i}$$

$$\psi^{(i)}(t) = 2^{i/2} g_1^{(i)}[k], \qquad \frac{k}{2^i} < t < \frac{k+1}{2^i}$$

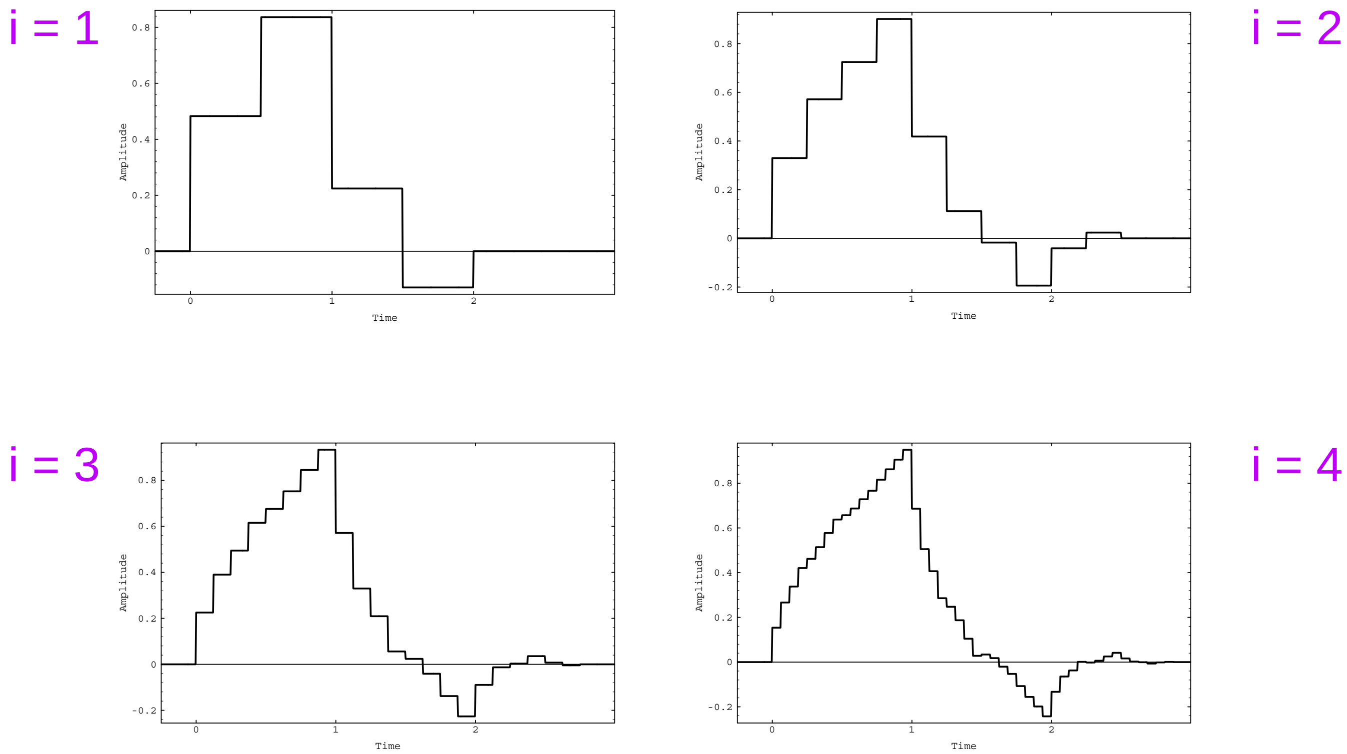
- this creates a graphical interpolation function
- iterated filters are of length ~ $2^i(L-1)$ given an initial filter of length L
- due to renormalization, the functions defined above will have finite length tending to $L-1$ as $i \to \infty$
- interesting convergence questions

Example:

$$G_0(z) = \frac{1}{\sqrt{2}} \cdot (1 + z^{-1}) \qquad \varphi^{(i)}(t) = I[0, 1]$$

Daubechies' system...
... example of iteration algorithm

] $$G_0(z) = \frac{1}{4\sqrt{2}}\left((1+\sqrt{3}) + z^{-1}(1-\sqrt{3}) + z^{-2}(3-\sqrt{3}) + z^{-3}(1-\sqrt{3})\right)$$

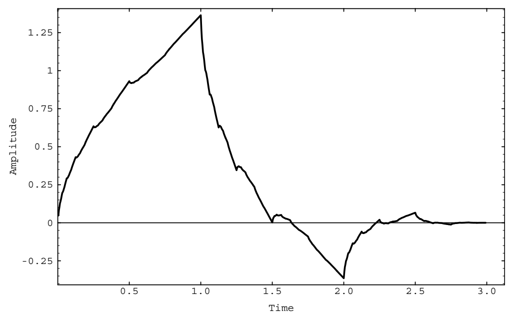


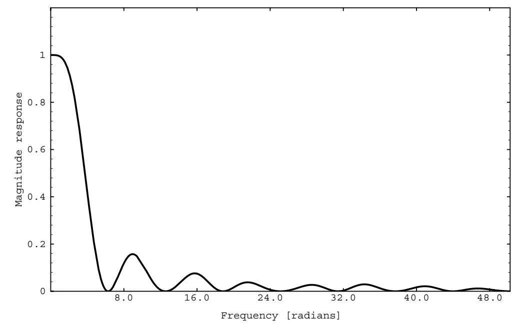
Fundamental link between discrete and continuous time!

Daubechies’ system...
... scaling function and wavelet

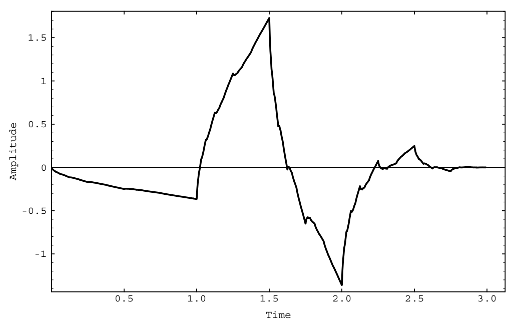
- Haar and sync systems: either good time OR frequency localization
- Daubechies system: good time AND frequency localization

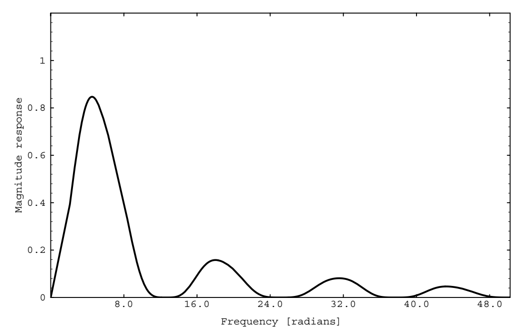
scaling function





wavelet



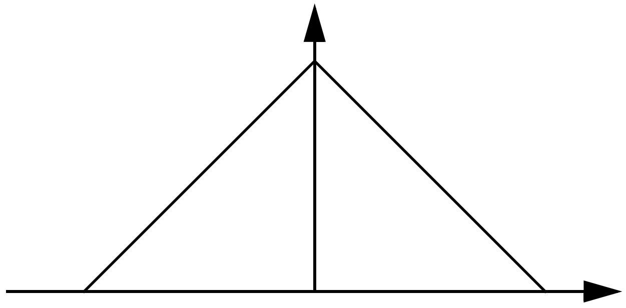


Finite length, continuous $\varphi(t)$ **and** $\psi(t)$**, based on** L=4 **iterated filter**

Daubechies' system...
... two-scale equation

$$\varphi(t) = \sum_n c_n \varphi(2t - n)$$

Hat function



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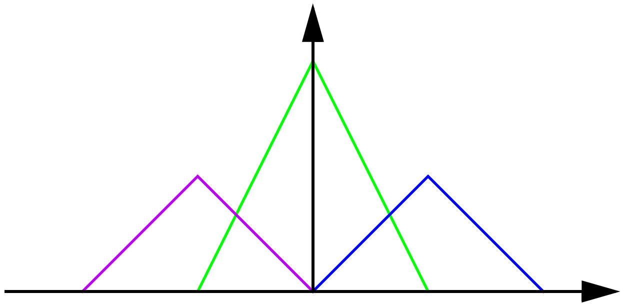
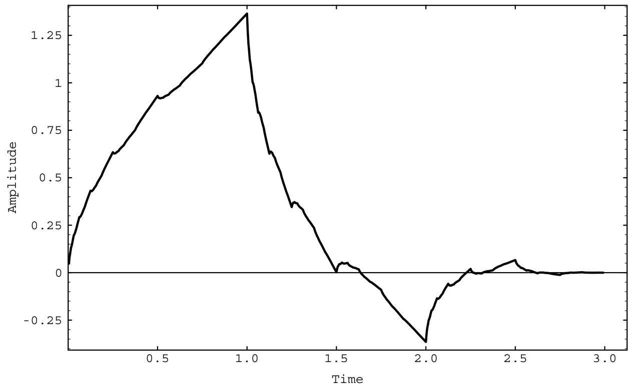


Daubechies' scaling function



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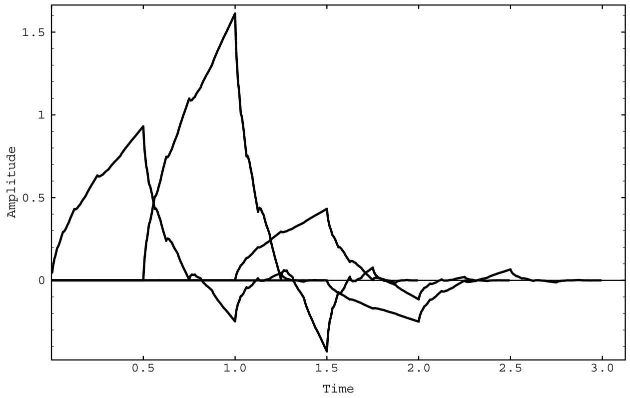


Daubechies' system...
... construction

- find an orthogonal filter of the form

$$G_0(z) = (1 + z^{-1})^N \cdot R(z)$$

- $R(z)$ is minimum degree polynomial such that

$$P(z) = G_0(z)G_0(z^{-1}) \text{ satisfies } P(z) + P(-z) = 2$$

- solve linear system for symmetric $Q(z)$ such that

$$P(z) = (1 + z)^N(1 + z^{-1})^N Q(z) \text{ satisfies above}$$

- if $Q(z)$ is positive definite, take spectral factorization (such as minimum phase)
- fortunately, for all N, solution possible!

Daubechies' system...
... example of construction

Example: N=2

$$P(z) = (1+z)^2(1+z^{-1})^2 Q(z) \text{ and } P(z)+P(-z) = 2 \text{ leads to}$$

$$Q(z) = -1/16\, z^{-1} + 1/4 - 1/16\, z$$

$$R(z) = \frac{1}{4\sqrt{2}}[1+\sqrt{3}+z^{-1}(1-\sqrt{3})]$$

$$G_0(z) = \frac{1}{4\sqrt{2}}[(1+\sqrt{3})+z^{-1}(1-\sqrt{3})+z^{-2}(3-\sqrt{3})+z^{-3}(1-\sqrt{3})]$$

Daubechies' system...
... family with increasing smoothness

- Daubechies' family is specified up to the "phase" of the spectral factor of $Q(z)$
- minimum-phase solutions are tabulated for $L = 4, 6, 8, 10, 12$ in Table 4.3, p. 260
- Hölder regularity estimates

N	L	$\alpha(N)$
2	4	0.5
3	6	0.915
4	8	1.275
5	10	1.596
6	12	1.888

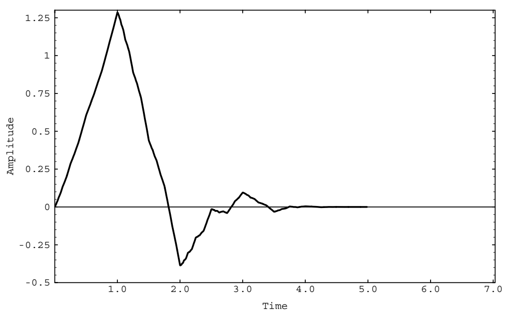
actually differentiable

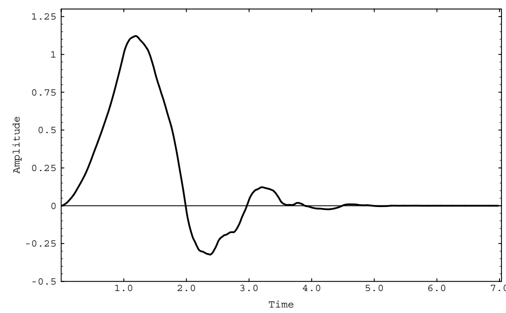
- closed-form formula for $Q(z)$

$$y = \cos\frac{\omega}{2}^2 \qquad P(y) = \sum_{i=0}^{N-1} \binom{N-1+j}{j} y^j$$

Daubechies' system...
... family with increasing smoothness

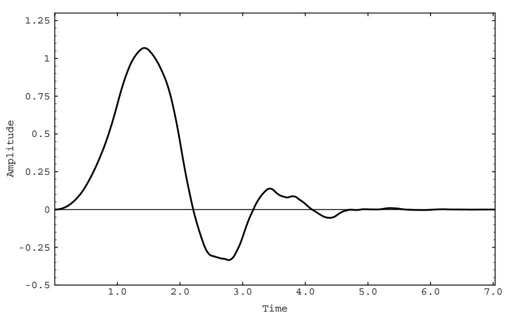
N = 3

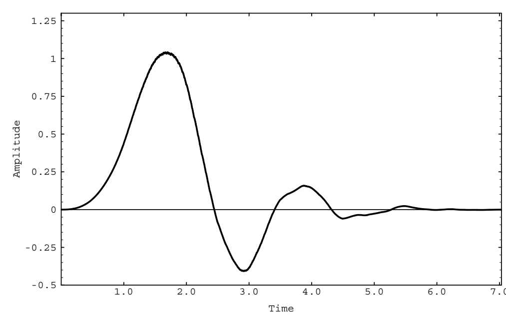




N = 4

N = 5





N = 6

Iterated filter banks...
... Fourier domain analysis

- assume convergence
- define $M_0(\omega) = 1/2^{1/2} G_0(e^{j\omega})$, or $M_0(1) = 1$
- then the graphical function becomes ($I(\omega)$: interpolation)

$$\Phi^{(i)}(\omega) = \left(\prod_{k=1}^{i-1} M_0\left(\frac{\omega}{2^k}\right) \right) \cdot I(\omega)$$

- the limit is

$$\Phi(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right) = M_0(\omega/2) \cdot \Phi(\omega/2)$$

- define $M_1(\omega) = 1/2^{1/2} G_1(e^{j\omega})$, then

$$\Psi(\omega) = M_1(\omega/2) \cdot \Psi(\omega/2)$$

A necessary condition for convergence is

$$M_0(\pi) = 0$$

Iterated filter banks...
... regularity analysis

- decompose $M_0(\omega) = 1/2(1+e^{j\omega})^N R(\omega)$, $R(\pi) \neq 0$
- consider infinite product

$$\Phi(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right) = \left(\prod_{k=1}^{\infty} \frac{1}{2}\left(1+e^{(j\omega)/2^k}\right)\right)^N \cdot \prod_{k=1}^{\infty} R\left(\frac{\omega}{2^k}\right)$$

- sufficient condition for continuity of $\varphi(t), \psi(t)$

$$\sup_{\omega \in [0, 2\pi]} |R(\omega)| < 2^{N-1}$$

Proof:

$1/2(1+e^{j\omega})$ goes to box function (Haar)

$1/2(1+e^{j\omega})^N 1/2(1+e^{j\omega})$ goes to B-spline of order N-1

thus, $1/\omega^N$ decay,

bound rest as not faster then $\omega^{N-1-\varepsilon}$

- products decays faster then $1/\omega$, proves continuity
- higher-order differentiability: similar argument

Iterated filter banks

Orthogonality relations

(a) $$\langle \varphi(t), \varphi(t-n) \rangle = \delta_n$$

(b) $$\langle \varphi(t), \psi(t-n) \rangle = 0$$

(c) $$\langle \psi(t), \psi(t-n) \rangle = \delta_n$$

With the definition

$$\psi_{mn}(t) = 2^{-m/2} \cdot \psi(2^{-m} t - n)$$

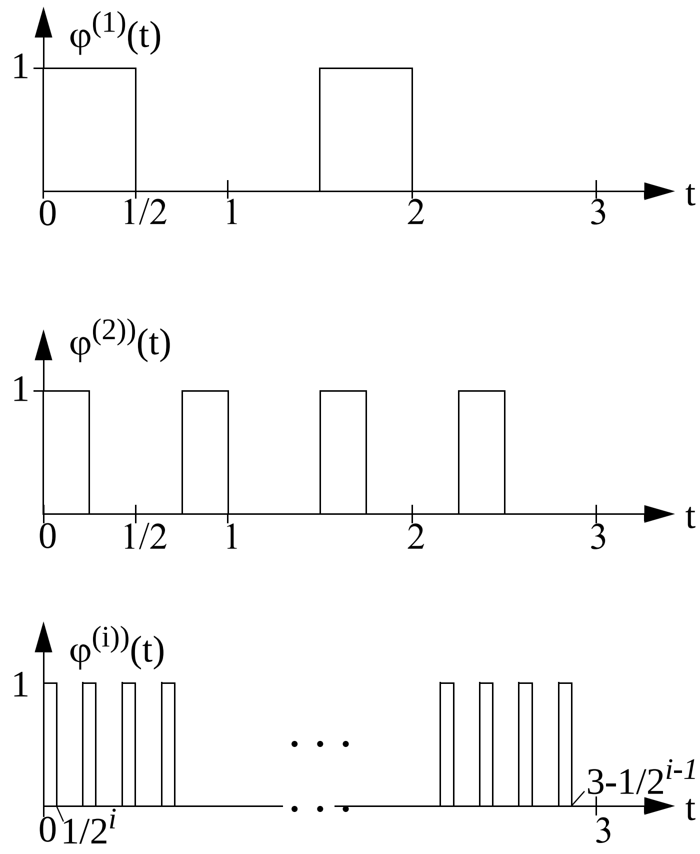
one can show

$$\langle \psi_{mn}(t), \psi_{kl}(t) \rangle = \delta_{mk} \cdot \delta_{nl}$$

and that $\{\psi_{mn}(t)\}_{mn}$ is an orthonormal basis for $L_2(\Re)$

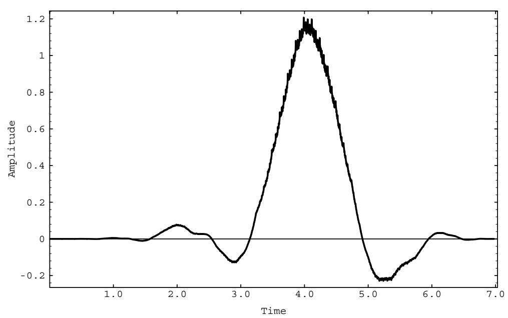
Iterated filter banks ...
... convergence issues

counter-example: $[1,0,0,1]/2^{1/2}$



absence of zero at π

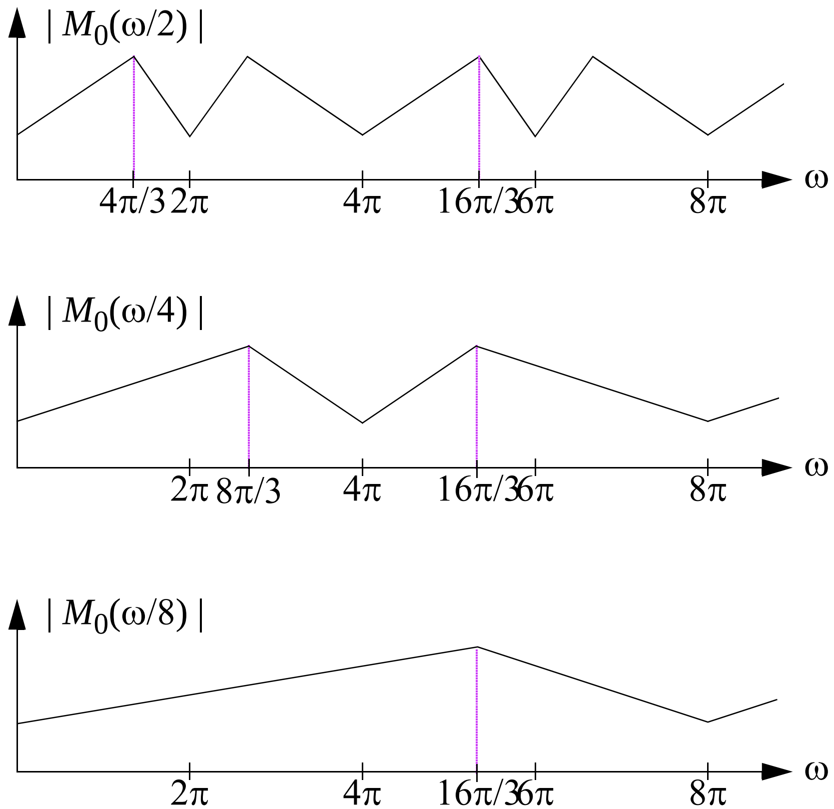
Smith & Barnwell, length 8



- one zero at π necessary for convergence

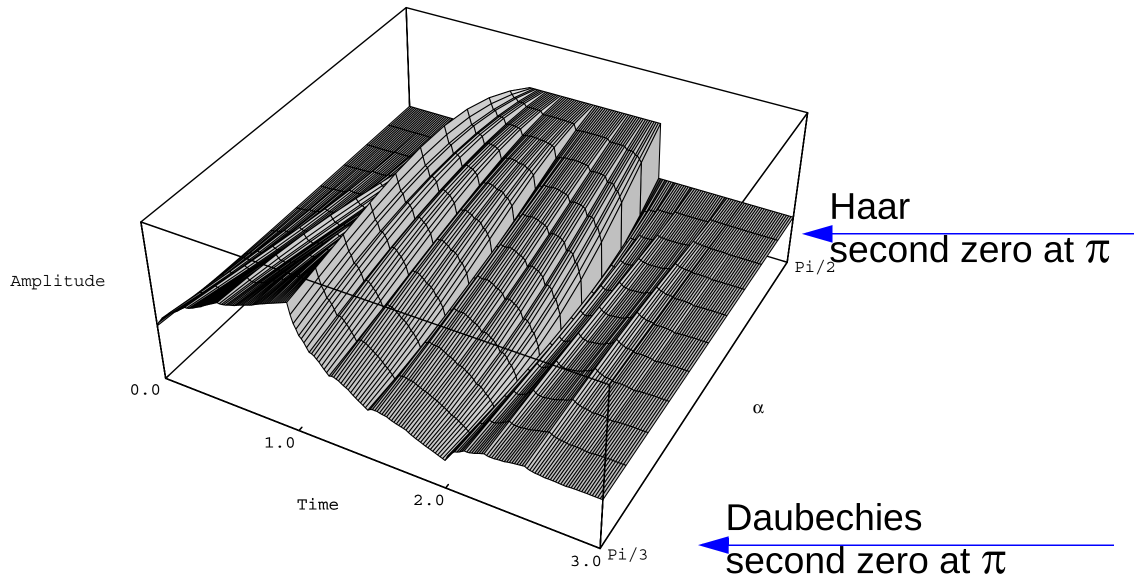
Iterated filter banks ...
... Cohen's method for investigating regularity

- fixed-point method, gives a lower bound



$$\prod_{k=1}^{i-1} \left|M_0(\omega/2^k)\right|_{\omega = (2^i\pi)/3} = \left|M_0(2\pi/3)\right|^i$$

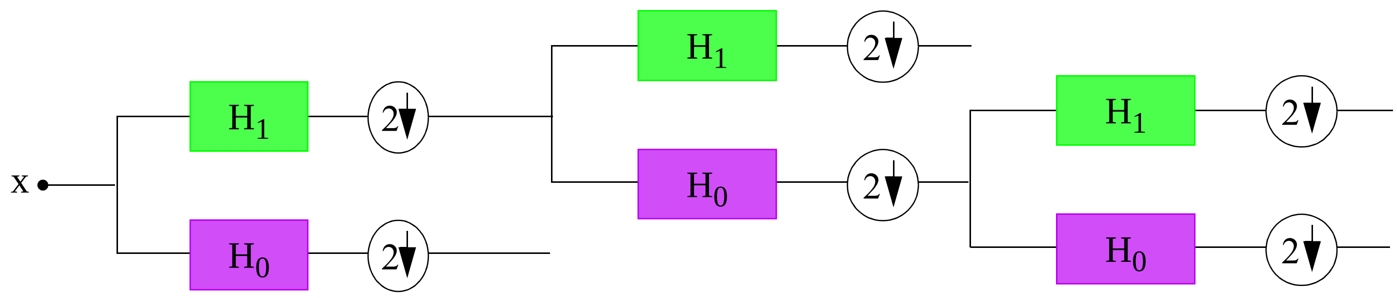
Iterated filter banks ...
... iteration of length-4 orthogonal family



- length 4, one zero at π
- sixth iteration

Wavelet packets

Iterate discrete-time filter banks, finish with arbitrary trees



- generates orthonormal basis for $l_2(Z)$ as well
- gives different frequency resolution
- consider equivalent basis sequences

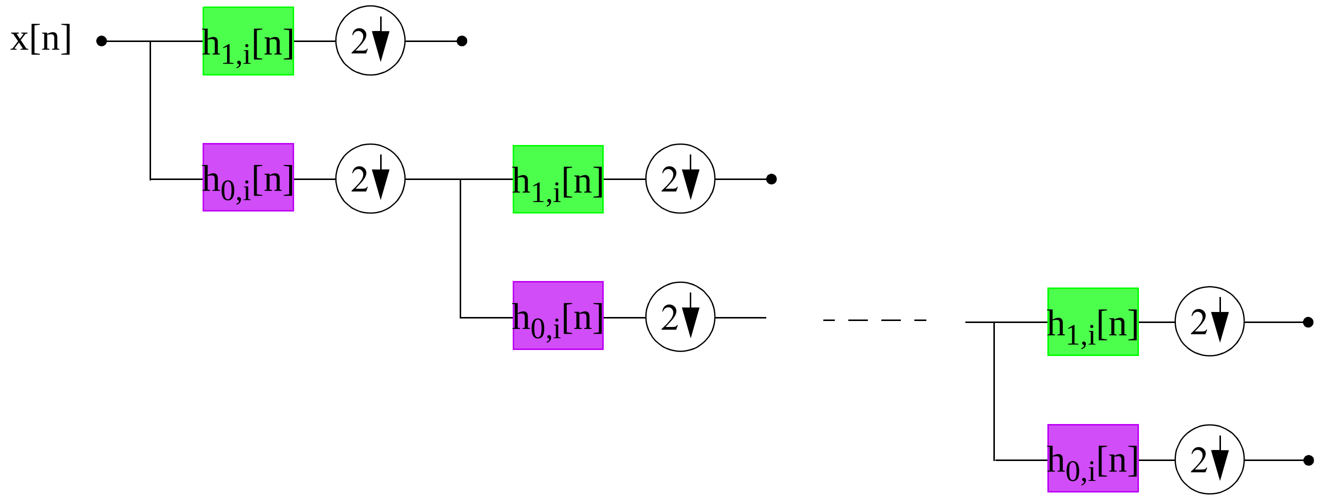
$$G_j^{(i)}(z) = \prod_{k=0}^{i-1} G_{0/1}\left(z^{2^k}\right)$$

- number of bases, depth-i tree

$$N(i) = N(i-1)^2 + 1$$

Multiwavelets

Iteration of a periodically time-varying filter



Matrix iteration but where $M_0(\omega)$ and $\Phi(\omega)$ are matrices

$$\Phi(\omega) = \prod_{k=1}^{\infty} M_0(\omega/2^k)$$

- several scaling functions, matrix two-scale equation
- necessary condition: eigenvalue/eigenvector condition
- continuous limits exist, several open questions

Wavelet series

Definition

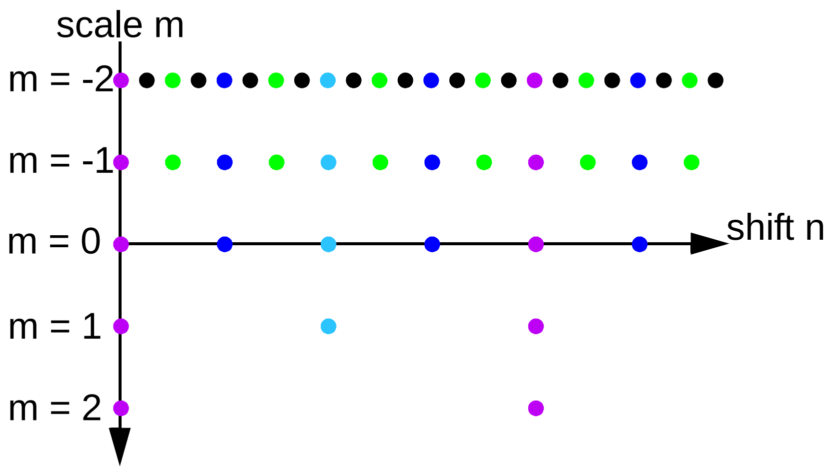
$$f(t) = \sum_{m,n} F[m,n]\psi_{mn}(t)$$

$$F[m,n] = \int f(t) \cdot \psi_{mn}(t)dt$$

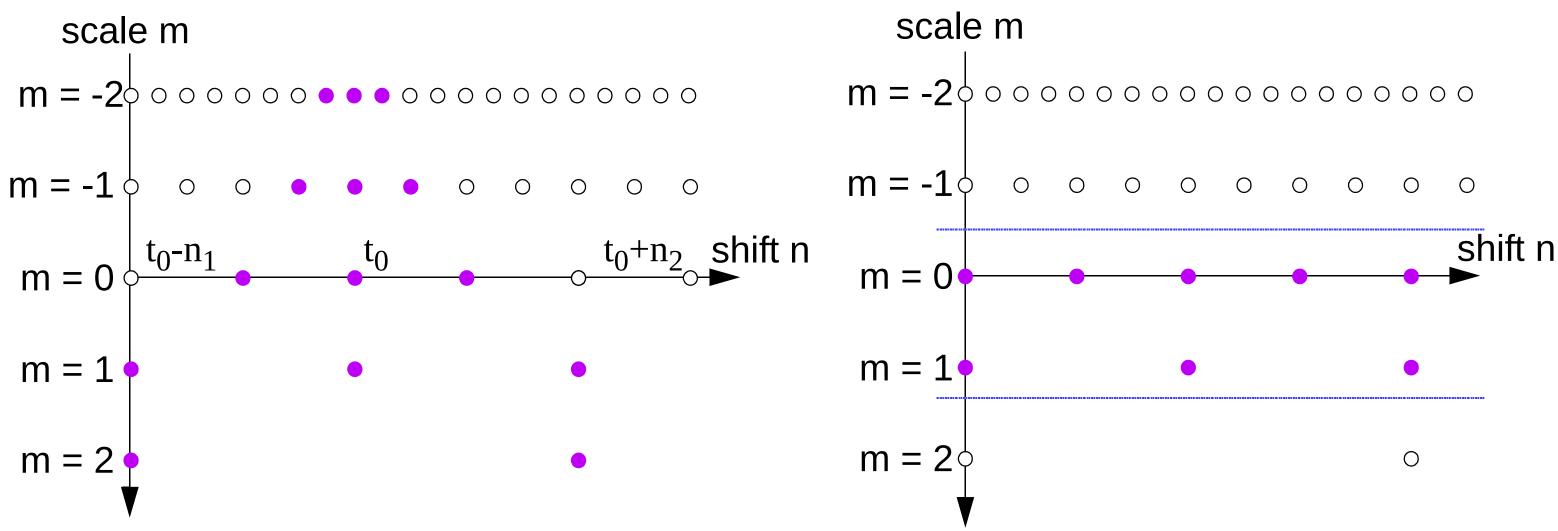
Properties

- linearity
- shift: in a weak sense
  (for scale-limited signals, by powers of 2)
- scaling: by powers of 2

Sampling of time-frequency plane



Wavelet series ...
... localization properties

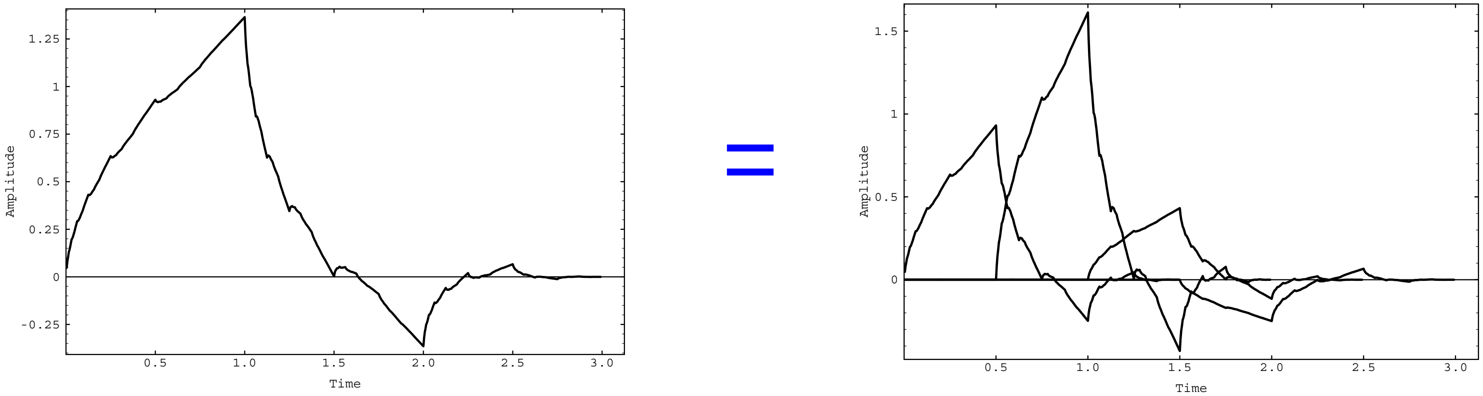


- time: exponential cone of influence
- frequency: octave-band resolution, local characterization of regularity
- pointwise characterization (see CWT as well)

Wavelet series...
... properties of basis functions

Two-scale equation

Daubechies’ wavelet



Moment properties of the wavelet

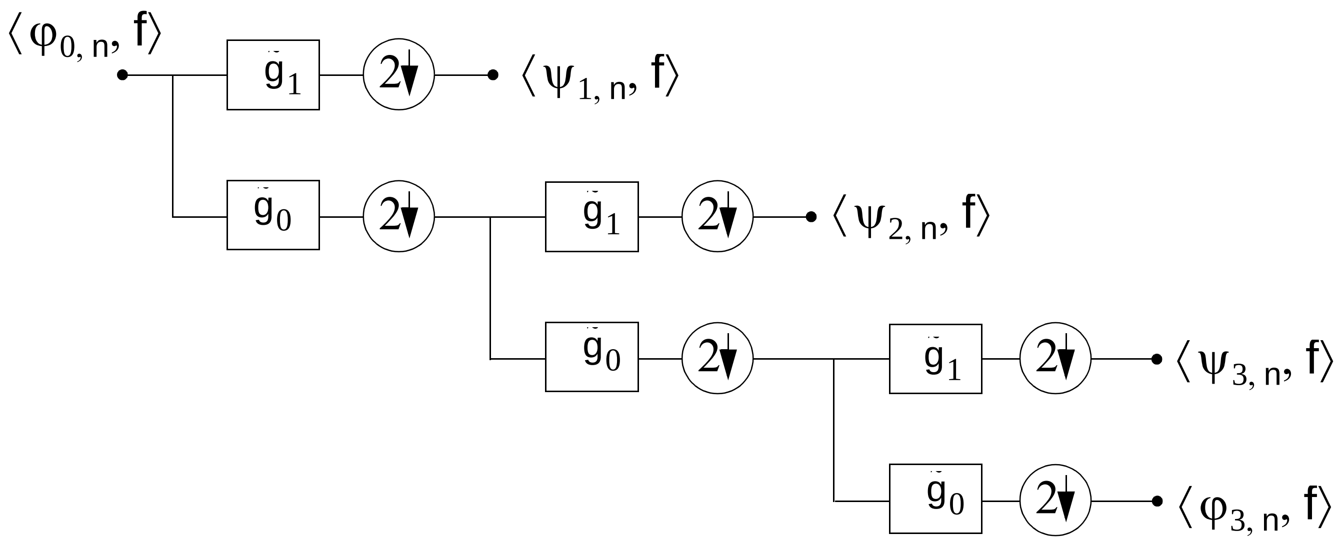
- if wavelet has N zero moments,
  then the first N terms of a Taylor series expansion are ‘‘killed’’: smooth functions are well compressed

Wavelet series ...
... computation using Mallat’s algorithm

- assume we start with an initial projection on V_0

$$f[n] = \langle \varphi_0(t-n), f(t) \rangle = \int \varphi_0(t-n) f(t) dt$$

- feed f[n] into filter bank algorithm to compute coefficients of wavelet series
- V_0 has to be ‘‘fine’’ enough (otherwise, choose V_{-i}, large i)



Generalizations of Wavelet Series

Biorthogonal wavelet series

Wavelet series based on recursive filters

Wavelet series based on multichannel filter banks

Multidimensional wavelet series

Biorthogonal wavelet series

Wavelet bass and dual basis

$$\langle \psi_{mn}(t), \psi_{kl}(t) \rangle = \delta[m-k]\delta[n-l]$$

Expansion in either basis

$$f(t) = \sum_{m,n} \tilde{F}[m,n]\psi_{mn}(t) = \sum_{m,n} F[m,n]\tilde{\psi}_{mn}(t)$$

Design based on biorthogonal filter banks

- $H_0(z)$, $H_1(z)$ when iterated lead to the dual basis
- $G_0(z)$, $G_1(z)$ when iterated lead to the basis

Difficult to get regular analysis and synthesis

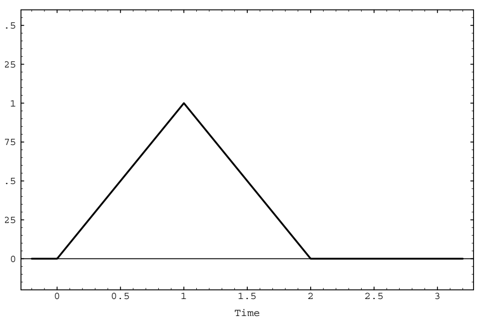
- synthesis needs to be regular

$$\tilde{\Phi}(\omega) = \prod_{k=1}^{\infty} \tilde{M}_0(\omega/2^k) \qquad \tilde{\Psi}(\omega) = \tilde{M}_1(\omega/2) \cdot \prod_{k=2}^{\infty} \tilde{M}_0(\omega/2^k)$$

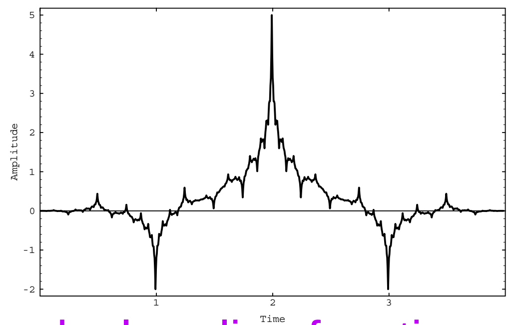
$$\Phi(\omega) = \prod_{k=1} M_0(\omega/2^k) \qquad \Psi(\omega) = M_1(\omega/2) \cdot \prod_{k=2} M_0(\omega/2^k)$$

Biorthogonal wavelet series ...
... example of a basis

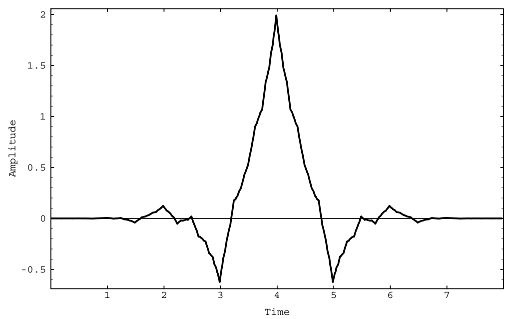
- $G_0(z) = 1/4 + 1/2\, z^{-1} + 1/4\, z^{-2}$ leads to hat function
- dual function based on length-5 filter, highly irregular
- dual function based on length-9 filter, regular



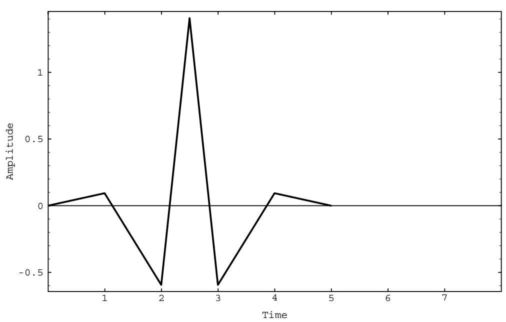
hat scaling fct



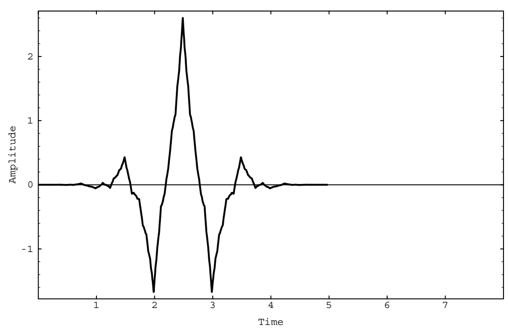
dual scaling function
length-5 filter



dual scaling function
length-9 filter

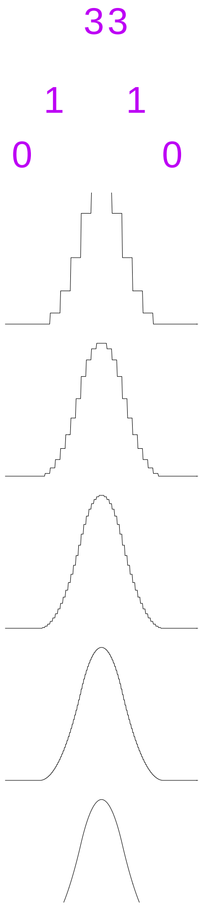


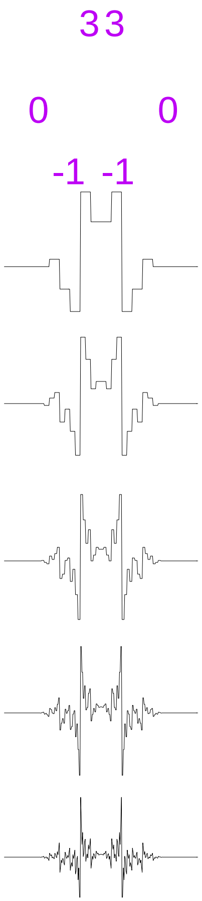
wavelet



dual wavelet

Biorthogonal wavelet series...
... iteration of two length-4 filters





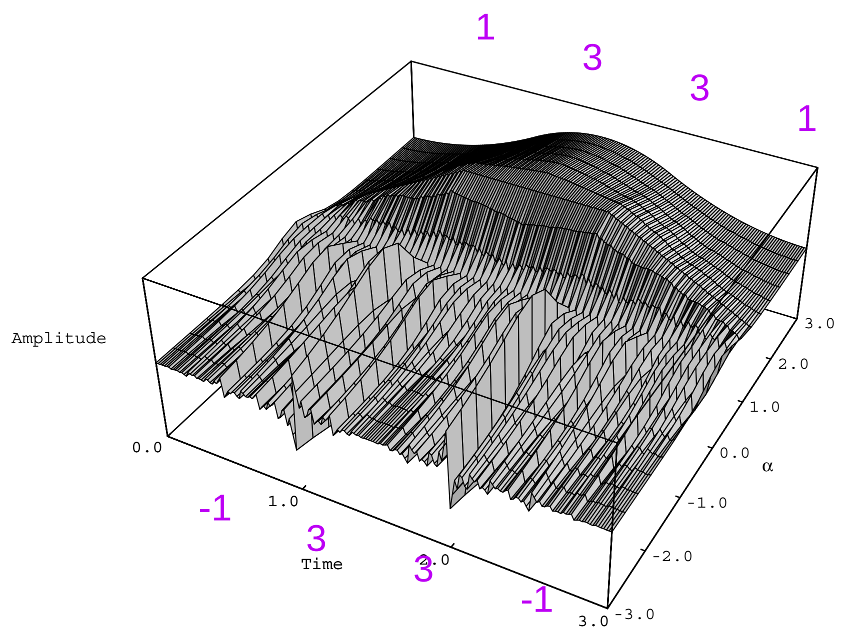
Biorthogonal wavelet series...
... iteration of length-4 biorthogonal family

- no regular analysis and synthesis
- filters

$$h_0 = C_h[1, \alpha, \alpha, 1]$$

$$g_0 = C_g[-1, \alpha, \alpha, -1]$$

lowpass filter with impulse response [1,α,α,1]



IIR wavelet series

Based on recursive (IIR) filters

- Daubechies: polynomial solution to maximally flat filters
- Herley: rational solutions as well

  Find

$$(1+z)^N \cdot (1+z^{-1})^N \cdot R(z) + (1-z)^N \cdot (1-z^{-1})^N \cdot R(-z) = 2$$

  such that $R(z)$ is all-pole or rational

- $P(z)$ has a special form, in particular, the Butterworth solution gives:

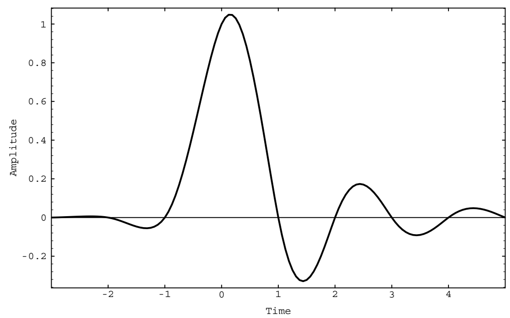
$$P(z) = \frac{2(1-z)^N \cdot (1-z^{-1})^N}{(z^{-1}+2+z)^N \cdot (-z^{-1}+2-z)^N}$$

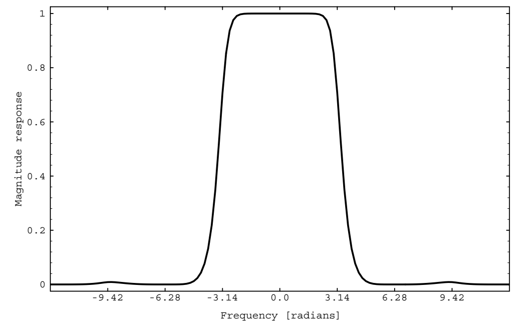
- such forms have closed form factorization (N odd)
- they are computationally more efficient and have better regularity than FIR-based wavelets

IIR wavelet series

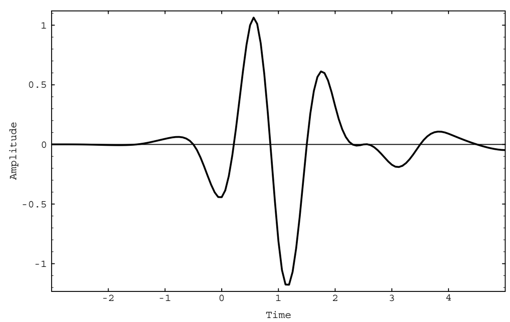
- best IIR solution: half-band Butterworth filters

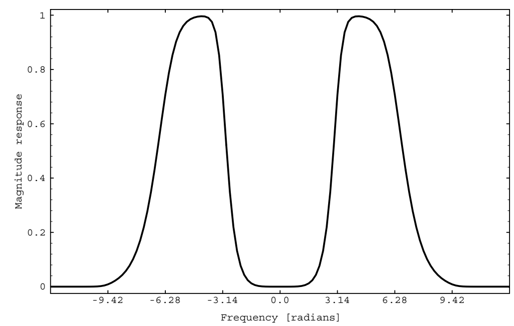
scaling function





wavelet





Wavelets series ...
... based on multichannel filter banks

Example: 4-channel filter bank based on depth-2 binary tree

- $F_0(z) = G_0(z)G_0(z^2)$ $\quad F_1(z) = G_0(z)G_1(z^2)$
- $F_2(z) = G_1(z)G_0(z^2)$ $\quad F_3(z) = G_1(z)G_1(z^2)$

- one scaling function with shift by 4
- three wavelets with scales by powers of 4

$$\{2^{-m} \cdot \Psi_i(4^{-m}t - n)\} \text{ with } i = 1, 2, 3$$

- this is an orthonormal basis for $L_2(\Re)$

General N-channel filter banks, downsampling by N

- scaling function

$$\Phi(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{N^k}\right)$$

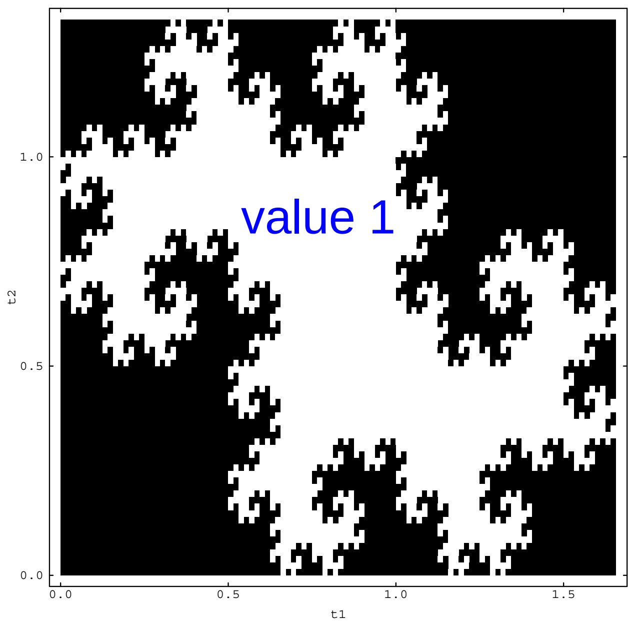
- regularity: sufficient number of zeros at the Nth roots of unity

$$M_0(\omega) = (1 + e^{j\omega} + \ldots + e^{(N-1)\omega}) \cdot R(\omega)$$

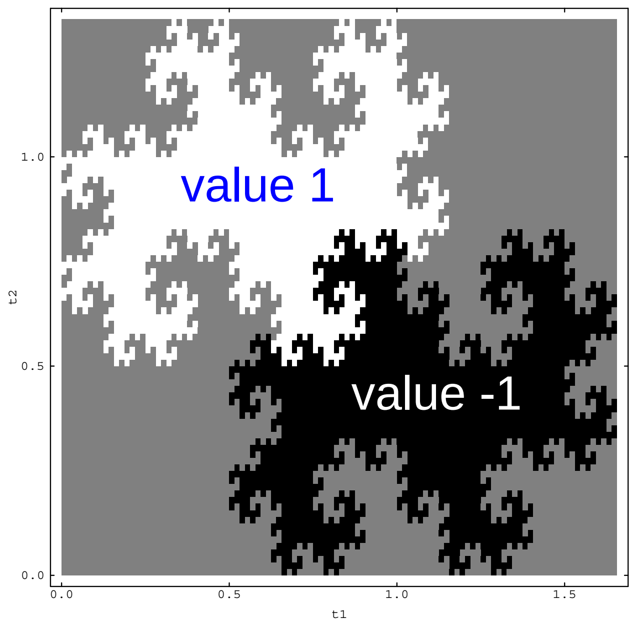
Wavelet series ...
... in multiple dimensions

Generalization of Haar

scaling function



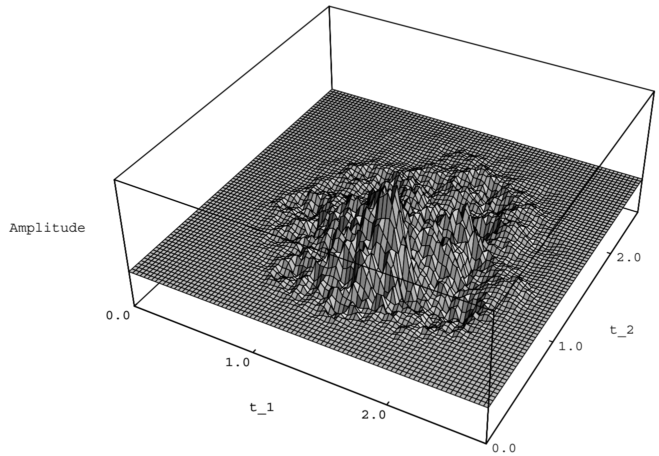
wavelet



“twin dragon” system

Wavelet series ...
... in multiple dimensions

Counterpart of Daubechies' wavelet [Kovacevic & Vetterli]

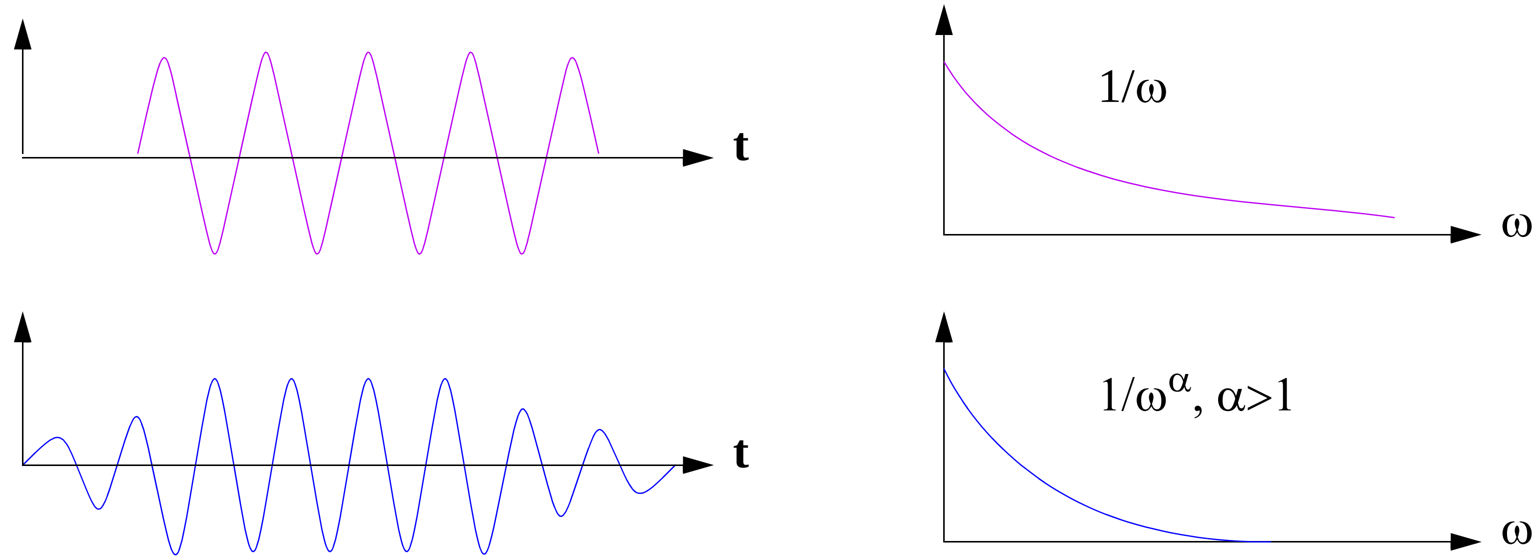


- smallest regular 2D wavelet - continuous
- quincunx lattice

Local cosine bases

Structured bases for $L_2(\Re)$: The Fourier case

- block Fourier transform: bad frequency localization
- Gabor transform: ill-behaved for critical sampling
- Balian-Low theorem: there is no local Fourier basis with good time and frequency localization
- however: good local cosine bases!



Note

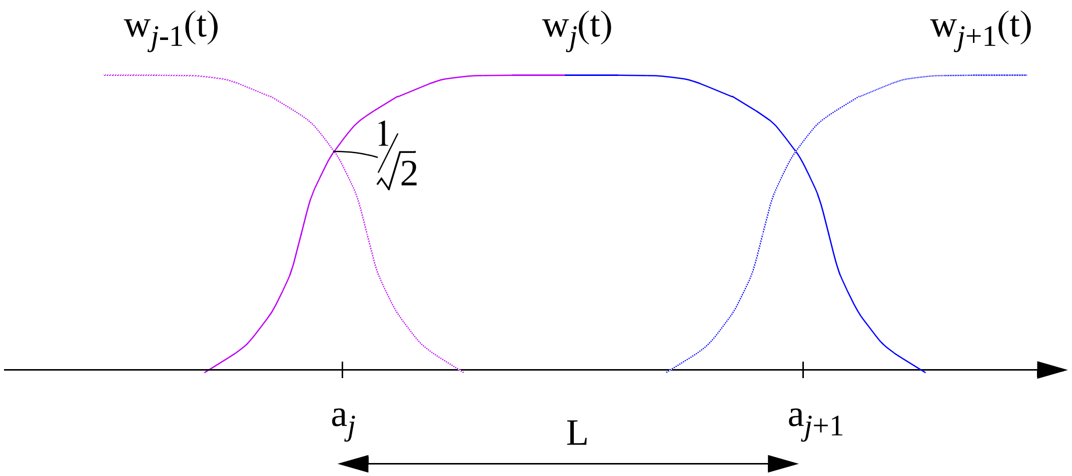
- shift and modulation

Local cosine bases

- continuous-time equivalent of LOT's
- shift by steps of size L
- modulation by cosine of frequency $\pi k/L$
- window function with power complementary property

$$\varphi_{j,k}(t) = \sqrt{2/L} \cdot \omega(t) \cdot \cos\left(\frac{\pi}{L}\left(k + \frac{1}{2}\right)\left(t - jL + \frac{L}{2}\right)\right)$$

- note: phase factor such that sine versus cosine behavior



- generalizes to windows of arbitrary length, as long as overlaps have proper symmetry